Linear Algebra

Problem Set 5 **Spring 2016**

Due Thursday, 21 April 2016 at 4:20 PM in EE105. This problem set covers Lectures 18-21. Free feel to work with others, but the final write-up should be entirely based on your own understanding. Be sure to print your name and student ID on your homework.

1. (20pts) In this problem, you are asked to interpret linear transformations geometrically.
   (a) Describe all linear transformations from $\mathbb{R}(=\mathbb{R}^1)$ to $\mathbb{R}$. What do their graphs look like?
   (b) Describe all linear transformations from $\mathbb{R}^2$ to $\mathbb{R}$. What do their graphs look like?
   (c) Consider the vector $\mathbf{v}=\begin{bmatrix}3\\2\\4\end{bmatrix}$. Is the transformation $T(\mathbf{x})=\mathbf{v}\cdot\mathbf{x}$ (the dot product) from $\mathbb{R}^3$ to $\mathbb{R}$ linear? If so, find the matrix of T.
   (d) The cross product of two vectors in $\mathbb{R}^3$ is defined by
$$\mathbf{x}\times\mathbf{y}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\times\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}x_2y_3-x_3y_2\\x_3y_1-x_1y_3\\x_1y_2-x_2y_1\end{bmatrix}.$$
   Is the transformation $T(\mathbf{x})=\mathbf{x}\times\mathbf{y}$ from $\mathbb{R}^3$ to $\mathbb{R}^3$ linear? If so, find the matrix of T in terms of the components of the vector $\mathbf{y}$.
2. (20pts) In this problem, you are asked to design geometric transformations.
   (a) Show that if T is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, then
$$T\left(\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}\right)=x_1T\left(\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}\right)+x_2T\left(\begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}\right)+\cdots+x_nT\left(\begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}\right).$$
   (b) Use (a) to find the matrix of the transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that rotates any vector through a given angle θ in the counterclockwise direction.
   (c) Find the matrix of orthogonal projection onto the line L in $\mathbb{R}^2$ that consists of all scalar multiples of the vector $\begin{bmatrix}3\\1\end{bmatrix}$.
   (d) Refer to (c). Find the matrix of reflection about the line L.

3. (20pts) Ture or false? Give a reason or counterexample. Let A be an m by n real matrix.
   (a) If A is one-to-one, then rank(A)=n.
   (b) If A is one-to-one, then A^T maps $\mathbb{R}^m$ onto $\mathbb{R}^n$.
   (c) If A maps $\mathbb{R}^n$ onto $\mathbb{R}^m$, then rank(A)=m.
   (d) If A is isomorphism, then m=n and A is nonsingular.
   (e) If A has linearly independent columns, then A is one-to-one.
   (f) If A has linearly independent rows, then A maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.
   (g) If A is one-to-one, then A^TA is isomorphism.
   (h) If A is one-to-one, then AA^T is isomorphism.
   (i) If A maps $\mathbb{R}^n$ onto $\mathbb{R}^m$, then A^TA is isomorphism.
   (j) If A maps $\mathbb{R}^n$ onto $\mathbb{R}^m$, then AA^T is isomorphism.
4. (25pts) Consider linear transformations in $\mathbb{R}^2$.
   (a) Consider the basis $\boldsymbol{\beta}=\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\2\end{bmatrix}\right\}$ of $\mathbb{R}^2$. If $\mathbf{x}=\begin{bmatrix}3\\-1\end{bmatrix}$, find $[\mathbf{x}]_{\boldsymbol{\beta}}$.
   (b) Refer to (a). Find the coordinate mapping T such that $T(\mathbf{x})=[\mathbf{x}]_{\boldsymbol{\beta}}$.
   (c) Refer to (a). Let $\boldsymbol{\gamma}=\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}\right\}$ be another basis of $\mathbb{R}^2$. Find a matrix P such that $[\mathbf{x}]_{\boldsymbol{\gamma}}=P[\mathbf{x}]_{\boldsymbol{\beta}}$.
   (d) Find a new basis $\boldsymbol{\beta}$ of $\mathbb{R}^2$ such that $\begin{bmatrix}1\\2\end{bmatrix}_{\boldsymbol{\beta}}=\begin{bmatrix}3\\5\end{bmatrix}$ and $\begin{bmatrix}3\\4\end{bmatrix}_{\boldsymbol{\beta}}=\begin{bmatrix}2\\3\end{bmatrix}$.
   (e) Find a new basis $\boldsymbol{\beta}$ of $\mathbb{R}^2$ such that the matrix representation of reflection T about the line in $\mathbb{R}^2$ spanned by $\begin{bmatrix}3\\1\end{bmatrix}$ with respect to $\boldsymbol{\beta}$ is diagonal. That is, $\left[T(\boldsymbol{\beta})\right]_{\boldsymbol{\beta}}$ is diagonal.
5. (15pts) Let $\boldsymbol{\beta}=\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ be a basis of $\mathbb{R}^3$ consisting of perpendicular unit vectors, such that $\mathbf{v}_3=\mathbf{v}_1\times\mathbf{v}_2$. Find the matrix representation of the given linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^3$ with respect to the basis $\boldsymbol{\beta}$.
   (a) $T(\mathbf{x})=\mathbf{x}-2(\mathbf{v}_3\cdot\mathbf{x})\mathbf{v}_3$
   (b) $T(\mathbf{x})=\mathbf{x}-2(\mathbf{v}_1\cdot\mathbf{x})\mathbf{v}_2$
   (c) $T(\mathbf{x})=\mathbf{v}_1\times\mathbf{x}+(\mathbf{v}_1\cdot\mathbf{x})\mathbf{v}_1$