Linear Algebra

Problem Set 6 **Spring 2013**

Due Tuesday, 7 May 2013 at 12:00 PM in EE208. This problem set covers Lecture 22-25. Free feel to work with others, but the final write-up should be entirely based on your own understanding. Be sure to print your name and student ID on your homework.

1. (25pts) Suppose T is a linear transformation, and

$$T(1,1)=\begin{bmatrix}1\\2\end{bmatrix},\ T(1,-1)=\begin{bmatrix}2\\3\end{bmatrix}.$$

Consider the following bases for $\mathbb{R}^2$:

$$S=\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\},\ B=\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\},\ C=\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}3\\1\end{bmatrix}\right\}.$$

Let $\mathbf{x}\in\mathbb{R}^2$. The coordinate vector of $\mathbf{x}$ relative to S is denoted by $[\mathbf{x}]_S$.

Similarly, we have $[\mathbf{x}]_B$ and $[\mathbf{x}]_C$. Let $[T]_S$ be the matrix representation of T with respect to S. Similarly, we have $[T]_B$ and $[T]_C$. Note that

$[T(\mathbf{x})]_S=[T]_S[\mathbf{x}]_S$.

(a) Find the change of coordinates matrices P and Q such that $[\mathbf{x}]_B=P[\mathbf{x}]_S$ and $[\mathbf{x}]_C=Q[\mathbf{x}]_S$.

(b) Find the change of coordinates matrices K such that $[\mathbf{x}]_C=K[\mathbf{x}]_B$.

(c) Given $[T]_S$, how can you find $[T]_B$? Write down the equation.

(d) Given $[T]_B$, how can you fin $[T]_C$? Write down the equation.

(e) Draw a diagram to illustrate the relationships among the following item:

$[\mathbf{x}]_S,[\mathbf{x}]_B,[\mathbf{x}]_c,[T]_S,[T]_B,[T]_C,P,Q,K.$

2. (15pts)

(a) If S is the subspace of $\mathbf{x}\in\mathbb{R}^3$ containing only the zero vector, what is $S^{\perp}$?

(b) If S is spanned by (1,1,1), what is $S^{\perp}$?

(c) If S is spanned by (1,1,1) and (1,1,0), what is $S^{\perp}$?

3. (10pts) Let A be an m by n matrix.

(a) Use the orthogonal subspaces theorem (2nd fundamental theorem of linear algebra) to explain why every $\mathbf{x} \in \mathbb{R}^n$ can be decomposed, in a unique way, into a row space component $\mathbf{x}_r \in C(A^T)$ and $\mathbf{x}_n \in N(A)$ such that $\mathbf{x} = \mathbf{x}_r + \mathbf{x}_n$.

(b) For every **b** in the column space of A, show that there is one and only one vector $\mathbf{x}_r \in C(A^T)$ satisfying $A\mathbf{x}_r = \mathbf{b}$. (Hint: See the last part of Lecture 23.)

4. (15pts)

(a) Suppose A is 3 by 4 and B is 4 by 5 and AB=0. Use the rank-nullity theorem to show that $\text{rank}(A) + \text{rank}(B) \leq 4$.

(b) Is it true that $x + y + z = 0$ and $x - 2y + z = 0$ are orthogonal subspaces because (1,1,1) is perpendicular to (1,-2,1)? Why or why not.

(c) Find a basis for the orthogonal complement of S, where S is spanned by (1,1,0,0) and (0,0,1,1). (Hint: Can you find a matrix so that its row space is S?)

5. (20pts) Let $A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix}$.

(a) Find the orthogonal projection matrix P onto the column space of A. Note that the columns of A are not linearly independent. So, you need to find a basis for the column space first. It is true that $C(P) = C(A)$?

(b) Find the point in the column space of A which is closest to (1,2,3).

(c) Find the orthogonal projection matrix Q onto the nullspace of A. There are two ways to do. The first way is to find the nullspace matrix of A, denoted by N. The column space of N is the nullspace of A. Then find the projection matrix onto the column space of N, as you did in (a). The second way is to find the projection matrix S onto the row space of A, which is the column space of A^T. Show that $Q = I - S$.

6. (15pts) There are two types of least-squares problems.

(a) Suppose $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find the least-squares solution $\hat{\mathbf{x}}$ so that $\|\mathbf{b} - A\hat{\mathbf{x}}\|^2$ is minimized.

(a) Suppose $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$. Find the solution of $A\mathbf{x} = \mathbf{b}$ so that $\|\mathbf{x}\|^2$ is minimized. Are you sure that such an **x** always in the row space of A?

(Hint: Write down $\mathbf{x} = \mathbf{x}_r + \mathbf{x}_n$. Then compute $\|\mathbf{x}\|^2$.)