Linear Algebra

Problem Set 6 **Spring 2015**

Due Thursday, 7 May 2015 at 4:20 PM in EE106. This problem set covers Lecture 22-25. Free feel to work with others, but the final write-up should be entirely based on your own understanding. Be sure to print your name and student ID on your homework.

1. (25pts) Suppose T is a linear transformation, and

$$T(1,1) = \begin{bmatrix} 3 \\ 7 \end{bmatrix},\ T(-1,1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Consider the following ordered bases for $\mathbb{R}^2$:

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\},\ B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\},\ C = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}.$$

Let $\mathbf{x} \in \mathbb{R}^2$. The coordinate vector of $\mathbf{x}$ relative to S is denoted by $[\mathbf{x}]_S$. Similarly, we have $[\mathbf{x}]_B$ and $[\mathbf{x}]_C$. Let $[T]_S$ be the matrix representation of T with respect to S. Similarly, we have $[T]_B$ and $[T]_C$. Note that $[T(\mathbf{x})]_S = [T]_S[\mathbf{x}]_S$.

(a) Find the change of coordinates matrices P and Q such that $[\mathbf{x}]_B = P[\mathbf{x}]_S$ and $[\mathbf{x}]_C = Q[\mathbf{x}]_S$.

(b) Find the change of coordinates matrices K such that $[\mathbf{x}]_C = K[\mathbf{x}]_B$.

(c) Given $[T]_S$, how can you find $[T]_C$? Write down the equation.

(d) Given $[T]_C$, how can you fin $[T]_B$? Write down the equation.

(e) Draw a diagram to illustrate the relationships among the following item: $[\mathbf{x}]_S, [\mathbf{x}]_B, [\mathbf{x}]_c, [T]_S, [T]_B, [T]_C, P, Q, K.$

2. (20ts)

(a) If S is spanned by (1,0,1,0) and (1,1,0,0), find a matrix A so that $S^{\perp} = C(A)$ and a matrix B so that $S = N(B)$.

(b) Will it be possible to find a matrix A so that $C(A) = N(A)^{\perp}$?

(c) Is it true that $x - y + z = 0$ and $x + 2y + z = 0$ are orthogonal subspaces because (1,-1,1) is perpendicular to (1,2,1)? Why or why not.

3. (15pts) Let A be an m by n matrix.
   (a) Use the orthogonal subspaces theorem (2^{nd} fundamental theorem of linear algebra) to explain why every $\mathbf{x} \in \mathbb{R}^n$ can be decomposed, in a unique way, into a row space component $\mathbf{x}_r \in C(A^T)$ and $\mathbf{x}_n \in N(A)$ such that $\mathbf{x} = \mathbf{x}_r + \mathbf{x}_n$.
   (b) For every **b** in the column space of A, show that there is one and only one vector $\mathbf{x}_r \in C(A^T)$ satisfying $A\mathbf{x}_r = \mathbf{b}$.
   (c) For every **b** in the column space of A, among all the solutions of $A\mathbf{x} = \mathbf{b}$, show that $\mathbf{x}_r \in C(A^T)$ minimizes $\|\mathbf{x}\|$. (Hint: compute $\|\mathbf{x}\|^2 = \|\mathbf{x}_r + \mathbf{x}_n\|^2$.)

4. (20pts) There are two types of least-squares problems. See (a) and (b).
   (a) Suppose $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. Find the least-squares solution $\hat{\mathbf{x}}$ so that $\|\mathbf{b} - A\hat{\mathbf{x}}\|^2$ is minimized.
   (b) Suppose $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Find the solution of $A\mathbf{x} = \mathbf{b}$ so that $\|\mathbf{x}\|^2$ is minimized.
   (c) Combine (a) and (b). Suppose $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Find the least-squares solution $\hat{\mathbf{x}}$ so that $\|\mathbf{b} - A\hat{\mathbf{x}}\|^2$ is minimized and $\|\mathbf{x}\|^2$ is minimized.

5. (20pts) Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.
   (a) Find the orthogonal projection matrix P onto the row space of A. Note that the rows of A are not linearly independent. So, you need to find a basis for the row space first. It is true that P and A have the same row space, i.e., $C(P^T) = C(A^T)$? Or, $C(P) = C(A^T)$?
   (b) Find the orthogonal projection matrix Q onto the nullspace of A.

(c) Find the point in the row space of A which is closest to (1,2,2).