Linear Algebra

Problem Set 6 **Spring 2016**

Due Thursday, 28 April 2016 at 4:20 PM in EE105. This problem set covers Lectures 22-25. Free feel to work with others, but the final write-up should be entirely based on your own understanding. Be sure to print your name and student ID on your homework.

1. (20pts) Let V be the vector space spanned by the functions e^x and e^{-x}, with the bases $\boldsymbol{\beta} = \{e^x, e^{-x}\}$ and $\boldsymbol{\gamma} = \{e^x + e^{-x}, e^x - e^{-x}\}$. Consider the linear transformation $D(f) = f'$ from V to V. For $f \in V$, the coordinate vector of f relative to $\boldsymbol{\beta}$ is denoted by $[f]_{\boldsymbol{\beta}}$.

   (a) Find the matrix representation of D with respect to $\boldsymbol{\beta}$, denoted by $[D]_{\boldsymbol{\beta}}$.

   (b) Find the change of coordinates matrices P and Q such that $[f]_{\boldsymbol{\beta}} = P[f]_{\boldsymbol{\gamma}}$ and $[f]_{\boldsymbol{\gamma}} = Q[f]_{\boldsymbol{\beta}}$.

   (c) Given $[D]_{\boldsymbol{\beta}}$, how can you find $[D]_{\boldsymbol{\gamma}}$? Write down the equation.

   (d) Draw a diagram to illustrate the relationships among the following item:
   $f, D, D(f), [f]_{\boldsymbol{\beta}}, [f]_{\boldsymbol{\gamma}}, [D]_{\boldsymbol{\beta}}, [D]_{\boldsymbol{\gamma}}, [D(f)]_{\boldsymbol{\beta}}, [D(f)]_{\boldsymbol{\gamma}}\ P, Q.$

2. (25pts) In this problem, you will study the sum of two subspaces.

   (a) Suppose S and T are two subspaces of a vector space V. The *sum* $S+T$ contains all sums $\mathbf{x}+\mathbf{y}$ of a vector $\mathbf{x}$ in S and a vector $\mathbf{y}$ in T. Show that $S+T$ is a subspace in V.

   (b) Explain the identity: $S + T = \text{span}\{S \cup T\}$.

   (c) Let $S^{\perp}$ denote the orthogonal complement of S. For every $\mathbf{v}$ in V, show that there exists a unique vector $\mathbf{x}$ in S and a unique vector $\mathbf{y}$ in $S^{\perp}$ such that $\mathbf{v} = \mathbf{x} + \mathbf{y}$. That is, $S + S^{\perp} = V$.

   (d) Prove the following identity:
   $$\dim(S+T) = \dim S + \dim T - \dim(S \cap T).$$
   See Strang's textbook (4th edition) pp 183 for a hint.

   (e) Show that $\dim S + \dim S^{\perp} = \dim V$.

3. (35pts) This problem is about orthogonal subspaces and orthogonal projection. Let

S be a subspace in $\mathbb{R}^4$ spanned by (1,0,1,0) and (1,0,0,1).

(a) Find a basis for $S^{\perp}$.

(b) Find a matrix A so that $S = N(A)$.

(c) Find the orthogonal projection matrix P onto S and the orthogonal projection matrix Q onto $S^{\perp}$.

(d) Show that $C(P) = S$ and $N(P) = S^{\perp}$.

(e) Given **v**=(1,1,1,1), find **x** in S and **y** in $S^{\perp}$ so that $\mathbf{v} = \mathbf{x} + \mathbf{y}$.

(f) Will it be possible to find a matrix B so that $C(B) = N(B)^{\perp} = S$? If yes, show the matrix B.

(g) If a 4 by 4 matrix M satisfies $M^T = M = M^2$, show that $\mathbf{x} - M\mathbf{x}$ is orthogonal to $M\mathbf{y}$ for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^4$.

4. (20pts) There are two types of least-squares problems. See (b) and (c).

(a) For every **b** in the column space of A, among all the solutions of $A\mathbf{x} = \mathbf{b}$, show that $\mathbf{x}_r \in C(A^T)$ minimizes $\|\mathbf{x}\|$. (Hint: compute $\|\mathbf{x}\|^2 = \|\mathbf{x}_r + \mathbf{x}_n\|^2$, where $\mathbf{x}_n \in N(A)$.)

(b) Suppose $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find the least-squares solution $\hat{\mathbf{x}}$ so that $\|\mathbf{b} - A\hat{\mathbf{x}}\|^2$ is minimized.

(c) Suppose $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$. Find the solution of $A\mathbf{x} = \mathbf{b}$ so that $\|\mathbf{x}\|^2$ is minimized.

(d) Combine (b) and (c). Suppose $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Find the least-squares solution $\hat{\mathbf{x}}$ so that $\|\mathbf{b} - A\hat{\mathbf{x}}\|^2$ is minimized and $\|\hat{\mathbf{x}}\|^2$ is minimized. (Hint: Project **b** onto $C(A)$.)