Linear Algebra

Problem Set 7 **Spring 2012**

Due Thursday, 3 May 2012 at 4:30 PM in EE208. This problem set covers Lecture 22-24. Free feel to work with others, but the final write-up should be entirely based on your own understanding. Be sure to print your name and student ID on your homework.

1. (30pts) Consider two bases $\mathcal{B}=\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$ and $\mathcal{C}=\left\{\begin{bmatrix}3\\2\\1\end{bmatrix},\begin{bmatrix}2\\-2\\4\end{bmatrix}\right\}$ for a subspace S in $\mathbb{R}^3$. Consider a linear transform $T:S\to S$, defined by $T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\\1\end{bmatrix}, T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)=\begin{bmatrix}2\\1\\1\end{bmatrix}$. Suppose the coordinate vector of **x** with respect to basis $\mathcal{B}$ is $[\mathbf{x}]_B=\begin{bmatrix}2\\3\end{bmatrix}$.
   (a) Find $T(\mathbf{x})$.
   (b) Find $[\mathbf{x}]_C$ and $[T(\mathbf{x})]_C$.
   (c) Find the change-of-coordinates matrix from basis $\mathcal{C}$ to $\mathcal{B}$, denoted by $C_{C\to B}$. Note that $[\mathbf{x}]_B = C_{C\to B}[\mathbf{x}]_C$.
   (d) Find the 2 by 2 matrix representation for T with respect to basis $\mathcal{B}$.
   (e) Find the 2 by 2 matrix representation for T with respect to basis $\mathcal{C}$.
   (f) Find the matrix A so that $[T(\mathbf{x})]_C = A[\mathbf{x}]_B$.

2. (20pts) 4.1, Exercise 30
   Suppoes A is 3 by 4 and B is 4 by 5 and AB=0. So $N(A)$ contains $C(B)$. Prove from the dimensions of $N(A)$ and $C(B)$ that rank$A$ + rank$B \le 4$.

3. (20pts) Suppose W is the subspace spanned by $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$.
   (a) Find the point in W that is closest to $\begin{bmatrix}1\\1\\1\end{bmatrix}$.

(b) Find the projection matrix P_W onto W. What is $\operatorname{rank} P_W$?

(c) Find the projection matrix $P_{W^\perp}$ onto $W^\perp$. What is $\operatorname{rank} P_{W^\perp}$?

(d) Find a basis for $W^\perp$. (Think of the column space of $P_{W^\perp}$.)

4. (15pts) Show that if $P^T = P$ and $P^2 = P$ then P is an orthogonal projection matrix. Note that P is an orthogonal projection matrix if $(I - P)\mathbf{x} \perp P\mathbf{y}$, for any **x** and **y**. This means that the column space of P is orthogonal to the column space of $(I - P)$.

5. (15pts) 4.2, Exercise 32
Suppose P_1 is the projection matrix onto the 1-dimensional subspace spanned by the first column of A. Suppose P_2 is the projection matrix onto the 2-dimensional column space of A. After thinking a little, compute the product P_2P_1.

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}.$$