Linear Algebra

Problem Set 7 **Spring 2013**

Due Tuesday, 14 May 2013 at 12:00 PM in EE208. This problem set covers Lecture 26-30. Free feel to work with others, but the final write-up should be entirely based on your own understanding. Be sure to print your name and student ID on your homework.

1. (15pts)
   (a) Express Gram-Schmidt orthogonalization process of
   $$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},\ \mathbf{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{a}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ as } A = QR.$$
   (b) Use the QR factorization to solve the least squares solution to $A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$.

2. (15pts) Suppose $Q = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix}$ is a 3 by 2 matrix with orthonormal columns.
   (a) Write down the orthogonal projection matrix onto the column space of Q.
   (b) Describe the nullspaces of Q, Q^T and QQ^T.
   (c) Suppose **a** is not in the column space of Q. Use Gram-Schmidt process to find $\mathbf{q}_3$ so that $\begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 \end{bmatrix}$ is an orthogonal matrix.
   (d) Suppose $\mathbf{b} = 2\mathbf{q}_1 + 3\mathbf{q}_2 + 4\mathbf{q}_3$. Find the least squares solution to $Q\mathbf{x} = \mathbf{b}$. What is the orthogonal projection of **b** onto the column space of Q?

3. (15pts) Let $H = I - 2\mathbf{u}\mathbf{u}^T$, where $\|\mathbf{u}\| = 1$.
   (a) What is H^{-1}? What is H^{100}?
   (b) Describe the set of **x** so that $H\mathbf{x} = \mathbf{x}$. Describe the set of **y** so that $H\mathbf{y} = -\mathbf{y}$.
   (c) Find **u** so that $H\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$.

4. (10pts) True or false, with a reason or counterexample if false. Suppose all the matrices are square.
   (a) $\det(A+B) = \det A + \det B$

(b) $\det(2A) = 2\det A$

(c) $\det(A^{10}) = (\det A)^{10}$

(d) If $A^T = -A$, then $\det A = 0$.

(e) If $A^T = A^{-1}$, then $\det A = 1$ or -1.

5. (15pts) It is known that $\begin{vmatrix} A & B \\ 0 & C \end{vmatrix} = (\det A)(\det C)$, where A and C are square.

(a) Let A and B be n by n. Derive a formula for $\begin{vmatrix} 0 & A \\ B & C \end{vmatrix}$.

(b) Let A, B, C, D be n by n. If D is invertible and $CD=DC$, show that $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \det(AD - BC)$. Hint: Consider $\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} I & 0 \\ -D^{-1}C & I \end{bmatrix}$.

(c) Are you sure that $\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 & 0 \\ a_{51} & a_{52} & 0 & 0 & 0 \end{vmatrix} = 0$? Why or why not.

6. (15pts) Use determinants to answer the following questions.

(a) A box has edges from (0,0,0) to (1,1,1) and (1,1,2) and (1,2,3). Find its volume.

(b) Find the area of the parallelogram with sides from (0,0,0) to (1,1,1) and (1,2,3).

(c) If $T(x, y) = \begin{bmatrix} 4x - 2y \\ 2x + 3y \end{bmatrix}$, find the area of $T(S)$, where $S = \{(x, y) | x^2 + y^2 \le 9\}$.

7. (15pts) Let $A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

(a) Use the Cramer's rule to compute the fourth column of A^{-1}.

(b) Find $\det C$, where C is the cofactor matrix of A. Hint: Use $A^{-1} = \frac{1}{\det A} C^T$.

(c) What is the cofactor matrix of C?