Linear Algebra

Problem Set 7 **Spring 2015**

Due Thursday, 14 May 2015 at 4:20 PM in EE106. This problem set covers Lecture 26-30. Free feel to work with others, but the final write-up should be entirely based on your own understanding. Be sure to print your name and student ID on your homework.

1. (15pts)
   (a) Express Gram-Schmidt orthogonalization process of
   $$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{a}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{a}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{as} \quad A = QR.$$
   (b) Use the QR factorization to solve the least squares solution to $A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

2. (20pts) Suppose $Q = [\mathbf{q}_1 \quad \mathbf{q}_2]$ is a 3 by 2 matrix with orthonormal columns.
   (a) Write down the orthogonal projection matrix onto the left nullspace of Q.
   (b) Describe the nullspaces of Q, Q^T and QQ^T.
   (c) Suppose $\mathbf{a}$ is not in the column space of Q. Use Gram-Schmidt process to find $\mathbf{q}_3$ so that $[\mathbf{q}_1 \quad \mathbf{q}_2 \quad \mathbf{q}_3]$ is an orthogonal matrix and the span of $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{a}\}$ is equal to the span of $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$.
   (d) Suppose $\mathbf{b} = \mathbf{q}_1 + 2\mathbf{q}_2 + 3\mathbf{q}_3$. Find the least squares solution to $Q\mathbf{x} = \mathbf{b}$. What is the orthogonal projection of $\mathbf{b}$ onto the left nullspace of Q?

3. (15pts) Let $H = I - 2\mathbf{u}\mathbf{u}^T$, where $\|\mathbf{u}\| = 1$.
   (a) What is H^{-1}? What is H^{99}?
   (b) Describe the set of $\mathbf{x}$ so that $H\mathbf{x} = \mathbf{x}$. Describe the set of $\mathbf{y}$ so that $H\mathbf{y} = -\mathbf{y}$.
   (c) Find $\mathbf{u}$ so that $H\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$.

4. (10pts) True or false, with a reason or counterexample if false. Suppose all the

matrices are square.

(a) $\det(-A) = -\det A$

(b) $\det(A^{100}) = (\det A)^{100}$

(c) If $A^T = -A$, then $\det A = 0$.

(d) If $A^T = A^{-1}$, then $\det A = 1$ or -1.

(e) If $A^2 = A$, then $\det A = 1$.

5. (10pts) It is known that $\begin{vmatrix} A & B \\ 0 & C \end{vmatrix} = (\det A)(\det C)$, where A and C are square.

(a) Let A and B be n by n. Derive a formula for $\begin{vmatrix} 0 & A \\ B & C \end{vmatrix}$.

(b) Let A and B be n by n. Show that $\begin{vmatrix} B & I \\ 0 & A \end{vmatrix} = \begin{vmatrix} B & I \\ -AB & 0 \end{vmatrix} = \det(AB)$. Thus,

$\det(AB) = (\det A)(\det B)$. Hint: Consider $\begin{bmatrix} B & I \\ -AB & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ -A & I \end{bmatrix}\begin{bmatrix} B & I \\ 0 & A \end{bmatrix}$.

6. (10pts) Use determinants to answer the following questions.

(a) A box has edges from (1,0,0) to (1,1,1) and (1,1,2) and (2,2,3). Find its volume.

(b) If $T(x, y) = \begin{bmatrix} x - 2y \\ -x + 5y \end{bmatrix}$, find the area of $T(S)$, where

$S = \{(x, y) \mid x^2 + y^2 \le 25\}$.

7. (20pts) Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}$.

(a) Use the Cramer's rule to compute the third column of A^{-1}.

(b) Find $\det C$, where C is the cofactor matrix of A. Hint: Use $A^{-1} = \frac{1}{\det A} C^T$.

(c) What is the cofactor matrix of C?

(d) What is the inverse of C?