Linear Algebra

Problem Set 7 **Spring 2016**

Due Tuesday, 10 May 2016 at 12:00 PM in EE105. This problem set covers Lectures 26-30. Free feel to work with others, but the final write-up should be entirely based on your own understanding. Be sure to print your name and student ID on your homework.

1. (10pts) If the $n \times n$ matrices A and B are orthogonal, which of the following matrices must be orthogonal as well?

   (a) $2A$ (b) $-A$ (c) A^2 (d) A^{-1} (e) A^T (f) AB (g) $A+B$ (h) $A^{-1}BA$

2. (10pts) Search for orthogonal matrices.

   (a) Find a 3×3 orthogonal matrix Q such that $Q\begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

   (b) Is there an orthogonal matrix Q such that $Q\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$ and

   $Q\begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$?

3. (30pts) Perform the Gram-Schmidt process, and thus find the QR factorization of a matrix.

   (a) Using paper an pencil, perform the Gram-Schmidt process on the sequences of vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}$.

   (b) Find the QR factorization of $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

   (c) Find the projection matrix P onto the column space of A. Hint: Use the QR factorization of A.

(d) Find the least-squares solutions of $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

(e) Find all vectors **b'**s so that the least-squares solutions of $A\mathbf{x} = \mathbf{b}$ is $\hat{\mathbf{x}} = \mathbf{0}$.

4. (10pts) True or false, with a reason or counterexample if false. Suppose A is an $n \times n$ real matrix.
   (a) If A is skew-symmetric, i.e., $A^T = -A$, then $\det A = 0$.
   (b) If A is an orthogonal matrix, then $\det A = 1$ or -1.
   (c) If A is a projection matrix, then $\det A = 1$ or 0.
   (d) If A is a nilpotent matrix, i.e., $A^k = 0$, for some positive integer k, then $\det A = 0$.
   (e) If $|\det A| = 1$, then the entries of A^{-1} are integers.

5. (20pts) Find the determinants of matrices.
   (a) Consider the matrix A_n defined as follows:

$$A_n = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 2 & 1 & \cdots & 0 \\ 0 & 0 & 1 & 2 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

   Express $\det A_n$ in terms of $\det A_{n-1}$ and $\det A_{n-2}$, and then find a closed formula for $\det A_n$.

   (b) Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 & 8 & 9 & 9 \\ 0 & 10 & 7 & 21 & 31 \\ 0 & 7 & 5 & 19 & 3 \\ 0 & 0 & 0 & 8 & 10 \\ 0 & 0 & 0 & 7 & 9 \end{bmatrix}.$$

   (c) Is it true that $\det \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = (\det A)(\det D)$? Explain. Note that A and D are square matrices.

   (d) If A, B, C, and D are $n \times n$ matrices, A is invertible and $AC=CA$, show that

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(AD - CB).$$

   Hint: Consider the product $\begin{bmatrix} I & 0 \\ -C & A \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}$.

6. (20pts) Use determinants to answer the following questions.

(a) Find the area of the parallelogram defined by the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(b) Find the area of a polygon with four vertices: (-7, 7), (-5, -6), (5, 5), (3, -4).

(c) If $T(x, y) = \begin{bmatrix} -x-6y \\ x+4y \end{bmatrix}$, find the area of $T(S) = \{T(\mathbf{x}) \mid \mathbf{x} \in S\}$, where $S = \{(x, y) \mid x^2 + y^2 \leq 1\}$.

(d) Refer to (c). Show that $T(S)$ is an ellipse.