Linear Algebra

Problem Set 8 **Spring 2015**

Due Thursday, 28 May 2015 at 4:20 PM in EE106. This problem set covers Lecture 32-35. Free feel to work with others, but the final write-up should be entirely based on your own understanding. Be sure to print your name and student ID on your homework.

1. (15pts) Let $A = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$. Find the eigenvalues and corresponding eigenvectors of A. What are the eigenvalues and corresponding eigenvectors of $A+3I$? What are the eigenvalues and corresponding eigenvectors of A^{10}?
2. (15pts) Find the eigenvalues of the following matrices.
   (a) $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ (Hint: what is the rank of A?)
   (b) $B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ (Hint: what is the relationship between A and B?)
   (c) $C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ (Hint: trace is helpful.)
3. (20pts) A 3 by 3 matrix A is known to have eigenvalues $0, 1, -1$. Use this information to find the values of the following items (give the answers where possible):
   (a) The rank of $A+I$
   (b) The eigenvalues of $A^T A$
   (c) The eigenvalues of $(A+2I)^{-1}$
   (d) The eigenvalues of $(A^2+I)^2$
   (e) The determinant of $3A-2I$
   (f) The determinant of $B^{-1}AB$
   (g) The determinant of A^2+A
   (h) The trace of $A+2I$
   (i) The trace of $3A+4A^T$

(j) The trace of $B^{-1}AB$

4. (20pts) True or false. Justify your answer. Suppose A is 3 by 3.

(a) If 0 is an eigenvalue of A, then A is singular.

(b) If $A^2=0$, then the only eigenvalue of A is 0.

(c) A nonzero vector cannot be the eigenvector corresponding to two different eigenvalues of A.

(d) If A has eigenvalues 0, 0, 0, then $A=0$.

(e) Each eigenvector of A is also an eigenvector of A^T.

(f) If A has distinct eigenvalues, then A is diagonalizable.

(g) If A has eigenvalues 0, 0, and 1, then the rank of A is 1.

(h) If A has eigenvalues 0, 0, 1, and A is not diagonalizable, then the rank of A is 2.

(i) If A has eigenvalues 1, 1, and 1, then A is not diagonalizable.

(j) If A is diagonalizable, then A^T is diagonalizable.

5. (20pts)

(a) Diagonalize $A=\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and derive the formula for A^k.

(b) The nth power of rotation through θ is rotation through $n\theta$:

$$R^n = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}.$$

Prove this formula by diagonalizing $R = S\Lambda S^{-1}$. You need to know Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.

(c) Substitute $A = S\Lambda S^{-1}$ into the product $(A-\lambda_1 I)(A-\lambda_2 I)\cdots(A-\lambda_n I)$ and explain why this produces the zero matrix. We are substituting the matrix A for the number λ in the polynomial $p(\lambda)=\det(A-\lambda I)$. The Cayley-Hamilton theorem says that this product is always $p(A)=$zero matrix, even if A is not diagonalizable. For example, if $A=\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, then $p(A)=(I-A)^2 = A^2-2A+I=0$.

(d) Use the Cayley-Hamilton theorem to compute A^{-1}, where $A=\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

6. (10pts) Suppose G_{k+2} is the average of the two previous numbers G_{k+1} and G_k. Let

$$\begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = A \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}.$$

(a) Find the limit as $k \to \infty$ of the matrices $A^k = S\Lambda^k S^{-1}$.

(b) If $G_0 = 0$ and $G_1 = 1$, show that the Gibonacci numbers approach $\frac{2}{3}$.