Linear Algebra

Problem Set 8 **Spring 2016**

Due Tuesday, 24 May 2016 at 12:00 PM in EE105. This problem set covers Lectures 32-35. Free feel to work with others, but the final write-up should be entirely based on your own understanding. Be sure to print your name and student ID on your homework.

1. (10%) For each of the following matrixes, find all eigenvalues and the corresponding (independent) eigenvectors.

   (a) $\begin{bmatrix} 4 & 5 \\ 0 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

2. (15pts) Let A be an invertible $n \times n$ matrix and $\mathbf{x}$ an eigenvector of A with associated eigenvalue λ.

   (a) Is $\mathbf{x}$ an eigenvector of A^2? If so, what is the eigenvalue?

   (b) Is $\mathbf{x}$ an eigenvector of A^{-1}? If so, what is the eigenvalue?

   (c) Is $\mathbf{x}$ an eigenvector of $A+2I$? If so, what is the eigenvalue?

   (d) Is $\mathbf{x}$ an eigenvector of $5A$? If so, what is the eigenvalue?

   (e) Is $\mathbf{x}$ an eigenvector of $A^2+2A+3I$? If so, what is the eigenvalue?

3. (30pts) True or false. Give a reason or counterexample.

   (a) If λ is an eigenvalue of A, then $N(A-\lambda I) \neq \{\mathbf{0}\}$.

   (b) If λ is an eigenvalue of A, then λ is an eigenvalue of A^T.

   (c) If $\mathbf{x}$ is an eigenvector of A with associated eigenvalue λ, then $\mathbf{x}$ is an eigenvector of A^T with associated eigenvalue λ.

   (d) If $\mathbf{x}$ is an eigenvector of both A and B, then $\mathbf{x}$ is an eigenvector of $A+B$.

   (e) If $\mathbf{x}$ is an eigenvector of both A and B, then $\mathbf{x}$ is an eigenvector of AB.

   (f) If A is singular, then 0 is an eigenvalue of A.

   (g) If $A^2=0$, then the only eigenvalue of A is 0.

   (h) If A is a 3×3 matrix and has eigenvalues 0, 1, and 2, then the rank of A is 2.

   (i) If A is a 3×3 matrix and has eigenvalues 0, 0, and 1, and A is not diagonalizable, then the rank of A is 2.

   (j) If A is a 3×3 matrix and has eigenvalues 0, 1, and 1, then the rank of A is 2.

4. (15pts) Arguing geometrically, find all eigenvectors and eigenvalues the linear transformations below.

   (a) Reflection about a line passing through (0,0) and (1,2) in $\mathbb{R}^2$.

   (b) Rotation through an angle of 180° in $\mathbb{R}^2$.

   (c) Orthogonal projection onto a line passing through (0,0) and (2,3) in $\mathbb{R}^2$.

5. (20pts) Let $A = \frac{1}{4}\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$.

(a) Diagonalize A.

(b) Find the formula for A^k.

(c) What is $\lim_{k\to\infty} A^k$?

(d) As $k \to \infty$, is it true that $A^k\mathbf{x}$ approaches an eigenvector of A for any nonzero vector $\mathbf{x}$?

6. (10pts) This problem is about Cayley-Hamilton theorem. Suppose A is diagonalizable.

(a) Substitute $A = S\Lambda S^{-1}$ into the product $(A-\lambda_1 I)(A-\lambda_2 I)\cdots(A-\lambda_n I)$ and explain why this produces the zero matrix. We are substituting the matrix A for the number λ in the polynomial $p(\lambda) = \det(A-\lambda I)$. The Cayley-Hamilton theorem says that this product is always $p(A) =$zero matrix, even if A is not diagonalizable.

(b) For a 3 by 3 matrix A, the Cayley-Hamilton theorem gives a matrix equation of the form $A^3 + a_2 A^2 + a_1 A + a_0 I = 0$. From this equation, you can get an expression of A^{-1}. Use the Cayley-Hamilton theorem to compute A^{-1}, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.