Linear Algebra

Problem Set 9 **Spring 2013**

Due Tuesday, 4 June 2013 at 12:00 PM in EE208. This problem set covers Lecture 35-37. Free feel to work with others, but the final write-up should be entirely based on your own understanding. Be sure to print your name and student ID on your homework.

1. (20pts) Let A be a 2 by 2 matrix with eigenvalues 3 and 1/3 and corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Let $\{\mathbf{u}_k,\ k = 0,1,2,\ldots\}$ be a solution of the difference equation $\mathbf{u}_{k+1} = A\mathbf{u}_k$, with $\mathbf{u}_0 = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$.
   (a) Compute $\mathbf{u}_1 = A\mathbf{u}_0$. (Hint: You do not need to know A itself.)
   (b) Find a formula for $\mathbf{u}_k$ in terms of k and the eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$.
   (c) Is this dynamical system stable, unstable or neutrally stable? Determine the nature of the origin (attractor, repellor, or saddle point) of the dynamical system $\mathbf{u}_{k+1} = A\mathbf{u}_k$.
   (d) Plot the trajectory of the dynamical system $\mathbf{u}_{k+1} = A\mathbf{u}_k$, with $\mathbf{u}_0 = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$.
2. (15pts) Let A be a 2 by 2 matrix with eigenvalues -1 and -3 and corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Let $\{\mathbf{u}(t),\ t \ge 0\}$ be a solution of the differential equation $\frac{d\mathbf{u}}{dt} = A\mathbf{u}(t)$, with $\mathbf{u}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.
   (a) Find a formula for $\mathbf{u}(t),\ t \ge 0$.
   (b) Is this dynamical system stable, unstable or neutrally stable?
3. (10pts) Section 6.3, Problem 6
   A has real eigenvalues but B has complex eigenvalues:
   $A = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}$ $B = \begin{bmatrix} b & -1 \\ 1 & b \end{bmatrix}$ (a and b are real). Find the conditions on a and b so that all solutions of $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$ and $\frac{d\mathbf{v}}{dt} = B\mathbf{v}$ approach zero as $t \to \infty$.
4. (15pts) Section 6.3, Problem 26
   Give two reasons why the matrix exponential e^{At} is never singular:
   (a) Write down its inverse.
   (b) Write down it eigenvalues. If $A\mathbf{x} = \lambda\mathbf{x}$ then $e^{At}\mathbf{x} = \underline{\quad}\ \mathbf{x}$.
5. (15pts) Section 6.3, Problem 31
   The cosine of a matrix is defined like e^A, by copying the series for cos t:

$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots \quad \cos A = I - \frac{A^2}{2!} + \frac{A^4}{4!} - \cdots$

(a) If $A\mathbf{x} = \lambda\mathbf{x}$, multiply each term times **x** to find the eigenvalue of cos *A*.

(b) Find the eigenvalues of $A = \begin{bmatrix} \pi & \pi \\ \pi & \pi \end{bmatrix}$ with eigenvectors (1,1) and (1,-1).

From the eigenvalues and eigenvectors of cos *A*, find that matrix $C = \cos A$.

6. (15pts) Find an invertible matrix *S* and a matrix *C* of the form $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that the given matrix has the form $A = \begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix} = SCS^{-1}$. Briefly explain the geometric interpretations of *A* using *S* and *C*.

7. (10pts) Let *A* be a real *n* by *n* matrix, and let **x** be a vector in $\mathbb{C}^n$. Show that

$\mathrm{Re}\{A\mathbf{x}\} = A\,\mathrm{Re}\{\mathbf{x}\}$ and $\mathrm{Im}\{A\mathbf{x}\} = A\,\mathrm{Im}\{\mathbf{x}\}$.