Linear Algebra

Problem Set 9 **Spring 2015**

Due Thursday, 4 June 2015 at 4:20 PM in EE106. This problem set covers Lecture 36-38. Free feel to work with others, but the final write-up should be entirely based on your own understanding. Be sure to print your name and student ID on your homework.

1. (15pts) Let $A=\begin{bmatrix}4 & 3\\ 0 & 1\end{bmatrix}$. Let $\{\mathbf{u}(t),\ t\geq 0\}$ be a solution of the differential equation $\frac{d\mathbf{u}}{dt}=A\mathbf{u}(t)$, with $\mathbf{u}(0)=\begin{bmatrix}5\\ -2\end{bmatrix}$.
   (a) Find a formula for $\mathbf{u}(t)=c_1e^{\lambda_1 t}\mathbf{x}_1+c_2e^{\lambda_2 t}\mathbf{x}_2,\ t\geq 0.$
   (b) Is this dynamical system stable, unstable or neutrally stable?
   (c) Find e^{At}.
2. (15pts)
   (a) Give two reasons why the matrix exponential e^{At} is never singular: write down its inverse and write down it eigenvalues. If $A\mathbf{x}=\lambda\mathbf{x}$ then $e^{At}\mathbf{x}=$ ____ $\mathbf{x}$.
   (b) When A is skew-symmetric ($A^T=-A$), $Q=e^{At}$ is orthogonal. Prove $Q^T=e^{-At}$ from the series for $Q=e^{At}$. Then show that $Q^TQ=I$.
3. (15pts) The cosine of a matrix is defined like e^A, by copying the series for cos t:
   $\cos t=1-\frac{t^2}{2!}+\frac{t^4}{4!}-\cdots \quad \cos A=I-\frac{A^2}{2!}+\frac{A^4}{4!}-\cdots$
   (a) If $A\mathbf{x}=\lambda\mathbf{x}$, multiply each term times $\mathbf{x}$ to find the eigenvalue of cos A.
   (b) If $A=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$, find that matrix $C=\cos A$.
4. (15pts)
   (a) Find an invertible matrix S and a matrix C of the form $C=\begin{bmatrix}a & -b\\ b & a\end{bmatrix}$ such that the given matrix has the form $A=\begin{bmatrix}1 & -2\\ 1 & 3\end{bmatrix}=SCS^{-1}$. Briefly explain the geometric interpretations of A using S and C.
   (b) Let $A=\begin{bmatrix}a & -b\\ b & a\end{bmatrix}$ (a and b are real). Find the conditions on a and b so that all solutions of $\frac{d\mathbf{u}}{dt}=A\mathbf{u}$ approach zero as $t\to\infty$.
5. (20pts) Show that A and B are similar by finding M so that $B=M^{-1}AM$.

(a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$.

6. (20pts) True or false, with a good reason.

   (a) A can't be similar to A^{-1} unless $A=I$.

   (b) A can't be similar to $A+I$.

   (c) If B is invertible, then AB is similar to BA.

   (d) If A is similar to B, then A^2 is similar to B^2.

   (e) A^2 and B^2 can be similar when A and B are not similar (try $\lambda = 0, 0$).