Linear Algebra

Problem Set 9 **Spring 2016**

Due Thursday, 2 June 2016 at 4:20 PM in EE105. This problem set covers Lectures 36-38. Free feel to work with others, but the final write-up should be entirely based on your own understanding. Be sure to print your name and student ID on your homework.

1. (15pts) A door is opened between rooms that hold $v(0) = 40$ people and $w(0) = 10$ people. The movement between rooms is proportional to the difference $v - w$:

$$\frac{dv}{dt} = w - v \quad \text{and} \quad \frac{dw}{dt} = v - w.$$

   (a) Show that the total is constant (50 people).

   (b) Find the matrix in $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$ and its eigenvalues and eigenvectors.

   (c) What are v and w at $t = 1$?

2. (10pts) When A is skew-symmetric ($A^T = -A$), $Q = e^{At}$ is orthogonal. Prove $Q^T = e^{-At}$ from the series for $Q = e^{At}$. Then $Q^TQ = I$.

3. (10pts) If $A^2 = A$, show that $e^{At} = I + (e^t - 1)A$. Use this result to compute e^{Bt}, where $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$.

4. (30pts) True or false. Give a reason or counterexample.

   (a) It is possible that A is similar to $A + I$.

   (b) If A is invertible and B is similar to A, then B is also invertible.

   (c) If A is similar to B, then A^2 is similar to B^2.

   (d) If A^2 is similar to B^2, then A is similar to B.

   (e) If A is invertible, then AB is similar to BA.

   (f) $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ is similar to $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

   (g) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is similar to $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

   (h) $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ is similar to $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

   (i) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is similar to $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

   (j) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is similar to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

5. (15pts) The cosine of a matrix is defined like e^A, by copying the series for $\cos t$:

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots \quad \cos A = I - \frac{A^2}{2!} + \frac{A^4}{4!} - \cdots$$

(a) If $A\mathbf{x} = \lambda\mathbf{x}$, multiply each term times $\mathbf{x}$ to find the eigenvalue of cos A.

(b) Find the eigenvalues of $A = \begin{bmatrix} \pi & \pi \\ \pi & \pi \end{bmatrix}$ with eigenvectors (1,1) and (1,-1).

From the eigenvalues and eigenvectors of cos A, find that matrix $C = \cos A$.

6. (10pts) Find an invertible matrix S and a matrix C of the form $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that the given matrix has the form $A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} = SCS^{-1}$.

7. (10pts) A has real eigenvalues but B has complex eigenvalues:

$A = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}$ $B = \begin{bmatrix} b & -1 \\ 1 & b \end{bmatrix}$ (a and b are real). Find the conditions on a and b so that all solutions of $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$ and $\frac{d\mathbf{v}}{dt} = B\mathbf{v}$ approach zero as $t \to \infty$.