Linear Algebra

Problem Set 1 Solution **Spring 2015**

1. (10pts)

$$\det(A) = 0 \;\Rightarrow\; \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 2 & 5 & 4 & b_2 \\ 3 & 7 & 6 & b_3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 1 & 0 & b_3 - 3b_1 \end{array}\right]$$

if $b_2 - 2b_1 \neq b_3 - 3b_1 \;\Rightarrow$ Ax = b have no solution.

if $b_2 - 2b_1 = b_3 - 3b_1 \;\Rightarrow$ Ax = b have infinite many solution.

$$b_2 - 2b_1 = b_3 - 3b_1 \;\Rightarrow\; b_1 + b_2 - b_3 = 0$$

if $b_1 + b_2 - b_3 = 0$, Ax = b have a solution.

2.(10pts)

$$\text{(a)} \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{(b)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix} \text{(c)} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{(d)} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

3.(10pts)

$$E_{21}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \Rightarrow E_{32}\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \Rightarrow E_{43}\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix}$$

$E_{43}\,E_{32}\,E_{21}\,$A = U.

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{2}{3} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{3}{4} & 1 \end{bmatrix},$$

$$U = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix}$$

4.(10pts)

$$\left[\begin{array}{ccc|c|c|c} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c|c|c} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c|c|c} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|c|c|c} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -4 & 1 & 4 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c|c|c} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -4 & 1 & 4 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 8 & -2 & -7 \\ 0 & 1 & 0 & -4 & 1 & 4 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

$$X = \begin{bmatrix} 8 & -2 & -7 \\ -4 & 1 & 4 \\ 1 & 0 & -1 \end{bmatrix}$$

5.(15pts)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 \text{ added to } r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{-r_3 \text{ added to } r_1} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 \text{ multiplied by -1}}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{-r_1 \text{ added to } r_3} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

6.(10pts)

(a)F

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 3 & 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad A \times B = \begin{bmatrix} 12 & 15 & 18 \\ 17 & 22 & 27 \\ 22 & 29 & 36 \end{bmatrix}$$

(b)T

(c)T

7.(15pts)

(a)

$$X = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & -2 & -3 & -4 \\ 1 & 1 & 2 & 2 & 2 \\ -1 & -2 & -2 & -3 & -2 \end{bmatrix}$$

(b)

$$X = \begin{bmatrix} 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

8.(20pts)

The same is true for all other columns of C.

$\because$ AC=I

$\therefore \mathrm{BAC} = \mathrm{B(AC)} = \mathrm{BI} = \mathrm{B}$

$\because$ BA=I

$\therefore \mathrm{BAC} = \mathrm{(BA)C} = \mathrm{IC} = \mathrm{C}$

$\Rightarrow \mathrm{BAC} = \mathrm{B(AC)} = \mathrm{(BA)C} = \mathrm{BI} = \mathrm{IC}$

$\Rightarrow$ left-inverse B equals the right-inverse C.