Linear Algebra

Problem Set 1 Solution **Spring 2016**

1. (15pts)

$$[A \quad |b] = \begin{bmatrix} 1 & 1 & -1 & | & 2 \\ 1 & 2 & 1 & | & 3 \\ 1 & 1 & k^2-5 & | & k \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & | & 2 \\ 0 & 1 & 2 & | & 1 \\ 0 & 0 & k^2-4 & | & -2+k \end{bmatrix}$$

(a) if $k \neq \pm 2$,Ax = b has a unique solution.

(b) if $k = 2 \rightarrow \begin{bmatrix} 1 & 1 & -1 & | & 2 \\ 0 & 1 & 2 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$,Ax = b has infinitely many solutions.

(c) if $k = -2 \rightarrow \begin{bmatrix} 1 & 1 & -1 & | & 2 \\ 0 & 1 & 2 & | & 1 \\ 0 & 0 & 0 & | & -4 \end{bmatrix}$,Ax = b has no solution.

2.(10pts)

$$\begin{cases} a+b+c = p_1 \\ a+2b+4c = p_2 \\ a+3b+9c = p_3 \end{cases} \rightarrow \begin{bmatrix} 1 & 1 & 1 & | & p_1 \\ 1 & 2 & 4 & | & p_2 \\ 1 & 3 & 9 & | & p_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & | & p_1 \\ 0 & 1 & 3 & | & -p_1+p_2 \\ 0 & 2 & 8 & | & -p_1+p_3 \end{bmatrix}$$

$\rightarrow \begin{bmatrix} 1 & 1 & 1 & | & p_1 \\ 0 & 1 & 3 & | & -p_1+p_2 \\ 0 & 0 & 2 & | & p_1-2p_2+p_3 \end{bmatrix}$, there exists an arbitrary polynomial for all values of $p1$, $p2$, $p3$.

3.(10pts)

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow[R_{13}^{(-7)}]{R_{12}^{(-4)}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \xrightarrow{R_{23}^{(-2)}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2^{(-\frac{1}{3})}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_{21}^{(-2)}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

It is impossible to transform $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ into $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

4.(10pts)

$[3 \quad a \quad b]^T = c_1[1 \quad 3 \quad 2]^T + c_2[2 \quad 6 \quad 4]^T + c_3[1 \quad 1 \quad 1]^T$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 3 & 6 & 1 & a \\ 2 & 4 & 1 & b \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 0 & -2 & -9+a \\ 0 & 0 & -1 & -6+b \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 3+a-2b \\ 0 & 0 & -1 & -6+b \end{array}\right]$$

$3 + \mathrm{a} - 2\mathrm{b} = 0 \to \mathrm{a} = 2\mathrm{b} - 3$, a and b are arbitrary constant.

5.(10pts)

$1：x ， $5：y ， $10：z

$$\begin{cases} x + y + z = 32 \\ x + 5y + 10z = 100 \end{cases} \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 32 \\ 1 & 5 & 10 & 100 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 32 \\ 0 & 4 & 9 & 68 \end{array}\right]$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c \begin{bmatrix} -5 \\ 9 \\ -4 \end{bmatrix} + \begin{bmatrix} 20 \\ 8 \\ 4 \end{bmatrix} \qquad (x, y, z) = (20,8,4)$$

6.(15pts)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_{12}^{(1)}} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_{21}^{(-1)}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_{12}^{(1)}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_{1}^{(-1)}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

7.(20pts)

(a)False

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 3 & 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, AB = \begin{bmatrix} 12 & 15 & 18 \\ 17 & 22 & 27 \\ 22 & 29 & 36 \end{bmatrix}$$

(b)True

$$Let\ BC = D, then\ \ AD = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} D = \begin{bmatrix} a_1 D \\ a_2 D \\ \vdots \\ a_n D \end{bmatrix} \quad ,\because a_2 = a_3 ,\therefore a_2 D = a_3 D$$

(c)False

$$Let\ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, A \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \to \begin{bmatrix} a+b & 2a+2b \\ c+d & 2c+2d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\begin{cases} c + d = 2 \\ 2c + 2d = 1 \end{cases}, there\ is\ no\ solution.$$

(d)True

$$\mathrm{Ax} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \ has\ a\ unique\ solution\ , so\ there\ exists\ A^{-1}.$$

$$then\ \ \mathrm{Ax} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \mathrm{x} = A^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \ also\ has\ a\ unique\ solution.$$

(e)False

$$Let\ A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , Ax = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} has\ infinitely\ many\ solutions.$$

$$But\ Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} has\ no\ solution.$$

8.(10pts)

(a) $X = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & -1 & 0 & -1 & -2 \\ 0 & 2 & 0 & 2 & 2 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$

(b) $X = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 5 & 1 & 6 & 2 \\ 9 & 1 & 10 & 2 \\ 13 & 1 & 14 & 2 \end{bmatrix}$