Solution to Problem 10

1.

$$A=\begin{bmatrix}2&2&2\\2&2&2\\2&2&2\end{bmatrix}\sim\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

(a)

$Consider\ the\ equation\ \ Ax=0=0\cdot x$

$\rightarrow nullity(A)=number\ of\ \ zero\ eigenvalues\ of\ \ A$

$\rightarrow \lambda_1=\lambda_2=0$

$Also, \sum_{i=1}^{i=3}\lambda_i=tr(A)\ \therefore \lambda_3=6$

(b)

$when\ \ \lambda_1=\lambda_2=0$

$\rightarrow (A-0I)x==0$

$$eigenvector\ \ \ x=\ span\{\begin{bmatrix}-1\\0\\1\end{bmatrix},\begin{bmatrix}-1\\1\\0\end{bmatrix}\}$$

$when\ \ \lambda_3=6$

$\rightarrow (A-6I)x=0$

$$eigenvector\ \ \ x=\ span\{\begin{bmatrix}1\\1\\1\end{bmatrix}\}$$

2.

(a)

$A\ doesn't\ have\ zero\ eigenvalue$

$\therefore nullity(A)=0$

$\therefore rank(A)=3-0=3$

(b)

$\det(A^TA)=\det(A^T)\det(A)=\det(A)^2=(1\cdot 2\cdot 3)^2=36$

(c)

We can't know that

(d)

We know that eigeneqaution is $(\lambda-1)(\lambda-2)(\lambda-3)=\lambda^3-6\lambda^2+11\lambda-6$

$\therefore tr(A)=6, tr(A+2I)=tr(A)+2tr(I)=12$

(e)

$\det((A^2+I)-\lambda I)=0$

We know that eigenvalue of A^2 *is* $1,4,9$

$\rightarrow$ *eigenvalue of* (A^2+I) *is* $2,5,10$

$\therefore$ *eigenvalue of* $(A^2+I)^{-1}$ *is* $0.5, 0.2, 0.1$

3

(a)

$$\det(A-\lambda I)=\det(\begin{bmatrix}3-\lambda & 2\\ -5 & -3-\lambda\end{bmatrix})=\lambda^2+1=0$$

From Cayley – Hamilton theory can know

$A^2+I=0$

$A^2=-I$

$A^4=I$

$\vdots$

$A^{1024}=A^{4*256}=I$

(b)

$$\det(B-\lambda I)=\det(\begin{bmatrix}5-\lambda & 7\\ -3 & -4-\lambda\end{bmatrix})=\lambda^2-\lambda+1=0$$

From Cayley – Hamilton theory can know

$B^2-B+I=0$

$B^2=B-I$

$B^4=B^2-2B+I=-B$

$\vdots$

$B^{1024}=B^{4*256}=-B$

4

(a)

$$\det(A-\lambda I)=\det(\begin{bmatrix} B-\lambda I & C \\ 0 & D-\lambda I \end{bmatrix})=0$$

$\rightarrow$ *eigenvalues of* $A=1,2,5,7$

(b)

$$\det(A-\lambda I)=\det(\begin{bmatrix} -\lambda & 1 & 1 & 1 \\ 1-\lambda & 1 & 1 & \\ 1 & 1-\lambda & 1 & \\ 1 & 1 & 1-\lambda & \end{bmatrix})=(\lambda+1)^3(\lambda-3)=0$$

$\lambda_1=\lambda_2=\lambda_3=-1,\lambda_4=3$

$\det(A)=-3$

5.

observe every term

$a_0=2$

$a_1=3$

$a_2=3\cdot 3-2\cdot 2=5$

$a_3=3\cdot 5-2\cdot 3=9$

$a_4=3\cdot 9-2\cdot 5=17$

$\vdots$

$a_k=2+2^k$

6.

(a)

Substitute $A = S\Lambda S^{-1}$ into $(A-\lambda I)$

$\to (A-\lambda I) = (S\Lambda S^{-1} - \lambda I) = S(\Lambda - \lambda I)S^{-1}$

$\to (A-\lambda_1 I)(A-\lambda_2 I)(A-\lambda_3 I)\ldots(A-\lambda_n I)$

$$= S\begin{bmatrix} 0 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} S^{-1} S \begin{bmatrix} \lambda_1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} S^{-1} \ldots S \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} S^{-1}$$

$= 0$

Then $p(A) =$ *zero matrix, which is the Cayley – Hamilton Theorem.*

(*If A is not diagonalizable, one proof is to take a sequence of diagonalizable matrices approaching A.*)

(b)

$$\det(A-\lambda I) = \begin{vmatrix} -3-\lambda & 2 & 0 \\ 2 & -3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$\lambda = 3, -5, 1$

Use Cayley – Hamilton theorem

$\to (A-3I)(A+5I)(A-I) = 0$

$\to A^3 + 3A^2 - 13A - 15I = 0$

$\therefore A^3 + 3A^2 - 13A - 17I$

$$= 0 - 2I = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$