2013 spring HW10

1.

(a) $B = M^{-1}AM \Rightarrow A = MBM^{-1}$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow \lambda_1 = 3\ ,\ \lambda_2 = 1 \Rightarrow x_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\ ,\ x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = MBM^{-1} \Rightarrow M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \Rightarrow \lambda_1 = 1\ ,\ \lambda_2 = 0 \Rightarrow x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\ ,\ x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = MBM^{-1} \Rightarrow M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

(c)

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \lambda_1 = 2\ ,\ \lambda_2 = 0 \Rightarrow x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\ ,\ x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \Rightarrow \lambda_1 = 2\ ,\ \lambda_2 = 0 \Rightarrow x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}\ ,\ x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow B = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\left(\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1} B \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\right)\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} B \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}^{-1}$$

$$\Rightarrow M = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

(d)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow \lambda^2 - 5\lambda - 2 = 0 \Rightarrow \lambda_1\ ,\ \lambda_2 \Rightarrow x_1 = \begin{bmatrix} 2 \\ \lambda_1 - 1 \end{bmatrix}\ ,\ x_2 = \begin{bmatrix} \lambda_2 - 4 \\ 3 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 2 & \lambda_2 - 4 \\ \lambda_1 - 1 & 3 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} 2 & \lambda_2 - 4 \\ \lambda_1 - 1 & 3 \end{bmatrix}^{-1}$$

$$B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \Rightarrow \lambda^2 - 5\lambda - 2 = 0 \Rightarrow \lambda_1\ ,\ \lambda_2 \Rightarrow x_1 = \begin{bmatrix} \lambda_1 - 1 \\ 2 \end{bmatrix}\ ,\ x_2 = \begin{bmatrix} 3 \\ \lambda_2 - 4 \end{bmatrix}$$

$$\Rightarrow B = \begin{bmatrix} \lambda_1 - 1 & 3 \\ 2 & \lambda_2 - 4 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} \lambda_1 - 1 & 3 \\ 2 & \lambda_2 - 4 \end{bmatrix}^{-1}$$

$$A = \begin{bmatrix} 2 & \lambda_2 - 4 \\ \lambda_1 - 1 & 3 \end{bmatrix}\left(\begin{bmatrix} \lambda_1 - 1 & 3 \\ 2 & \lambda_2 - 4 \end{bmatrix}^{-1} B \begin{bmatrix} \lambda_1 - 1 & 3 \\ 2 & \lambda_2 - 4 \end{bmatrix}\right)\begin{bmatrix} 2 & \lambda_2 - 4 \\ \lambda_1 - 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} B \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1}$$

$$\Rightarrow M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

2.

similar matrices have the same eigenvalues.

$\Rightarrow$ The matrices have different eigenvalues are not similar.

$< \text{case1}: \lambda = 0, 0 >$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ this matrix is the only one in the family.

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$,they are in the same family.

$< \text{case2}: \lambda = 1, 0 >$

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ they are in the same family.

$< \text{case3}: \lambda = 1, 1 >$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ this matrix is the only one in the family.

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ they are in the same family.

$< \text{case4}: \lambda = 1, -1 >$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$< \text{case5}: \lambda = 2, 0 >$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$< \text{case6}: \lambda = \dfrac{1 \pm \sqrt{5}}{2} >$

$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ they are in the same family.

$\Rightarrow$ There are 8 families.

3.

(a)False

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is similar to $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

(b)True

Invertible matrix doesn't have $\lambda = 0$, but sigular matrix has $\lambda = 0$.

$\Rightarrow$ The matrices have different eigenvalues are not similar.

(c)False

$A=\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ is similar to $-A=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(d)True

eigenvalues of $A \neq$ eigenvalues of $A+I$

(e)True

$AB = B^{-1}(BA)B$

(f)True

$A = MBM^{-1} \Rightarrow A^2 = (MBM^{-1})(MBM^{-1}) = MB^2M^{-1}$

(g)True

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $A^2 = B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

A^2 are similar to B^2, but A is not similar to B.

4.

(a)

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \Rightarrow \lambda_1 = 2 \text{ , } \lambda_2 = 4 \Rightarrow x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ , } x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow q_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ , } q_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A = 2\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \end{bmatrix}\right) + 4\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \end{bmatrix}\right)$$

$$B = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \Rightarrow \lambda_1 = 0 \text{ , } \lambda_2 = 25 \Rightarrow x_1 = \begin{bmatrix} 4 \\ -3 \end{bmatrix} \text{ , } x_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \Rightarrow q_1 = \frac{1}{5}\begin{bmatrix} 4 \\ -3 \end{bmatrix} \text{ , } q_2 = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$B = 25\left(\frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)\left(\frac{1}{5}\begin{bmatrix} 3 & 4 \end{bmatrix}\right)$$

(b)

$$P_1 + P_2 = q_1q_1^T + q_2q_2^T = \begin{bmatrix} q_1 & q_2 \end{bmatrix}\begin{bmatrix} q_1^T \\ q_2^T \end{bmatrix} = QQ^T = I$$

$$P_1P_2 = q_1q_1^Tq_2q_2^T = q_1\left(q_1^Tq_2\right)q_2^T = 0 \;\; (\because q_1 \perp q_2)$$

5.

(a)False

$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ eigenvalues are 1,1 ; eigenvector is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$; but it is not symmetric.

(b)True

$A = Q\Lambda Q^{-1} = Q\Lambda Q^{T}$; $A^{T} = \left(Q\Lambda Q^{T}\right)^{T} = Q\Lambda^{T}Q^{T} = Q\Lambda Q^{T} = A$

(c)True

$A = A^{T}$; $A^{-1} = \left(A^{T}\right)^{-1} = \left(A^{-1}\right)^{T}$

(d)False

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \lambda_1 = 2$, $\lambda_2 = 0 \Rightarrow x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ which is not symmetric.

6.

$\det(A_1) = c > 0$

$\det(A_2) = c^2 - 1 > 0 \Rightarrow c > 1$ or $c < -1$

$\det(A_3) = c^3 - 3c + 2 = (c-1)^2(c+2) > 0 \Rightarrow c > -2$

$\Rightarrow$ If $c > 1$, it is positive definite.

$\det(B_1) = 1 > 0$

$\det(B_2) = d - 4 > 0 \Rightarrow d > 4$

$\det(B_3) = 12 - 4d > 0 \Rightarrow d < 3$

$\Rightarrow d$ doesn't exist. $\Rightarrow$ it isn't positive definite.

7.

(a)

A is positive deinite $\Leftrightarrow$ All n eigenvalues of A are positive, $\lambda_i > 0$

$\det(A) = \lambda_1\lambda_2 \ldots\ldots \lambda_n > 0 \Rightarrow$ A is invertible.

(b)

A is positive deinite if $x^{T}Ax > 0$ for all $x \neq 0$. Here we choose $x = \begin{bmatrix} 0 & 0 & \cdots & 1 & \cdots & 0 \end{bmatrix}^{T}$

diagonal entries $= \begin{bmatrix} 0 & 0 & \cdots & 1 & \cdots & 0 \end{bmatrix} A \begin{bmatrix} 0 & 0 & \cdots & 1 & \cdots & 0 \end{bmatrix}^{T} > 0$

(c)

All projection matrice are singular except I.

(d)

All diagonal entries are eigenvalues. $\because$ diagonal entries are positive. $\therefore \lambda_i > 0$

All n eigenvalues of A are positive, $\lambda_i > 0$ $\Leftrightarrow$ A is positive deinite.

(e)

$\det(A) = \lambda_1\lambda_2 \ldots\ldots \lambda_n > 0$, but λ_i might be negative. $\Rightarrow$ It might not be positive deinite.