Linear Algebra

Problem Set 10 Solution **Spring 2015**

1.(10pts)

Let det(A- λ I) = 0 $\Rightarrow$ λ = 0, 3, -3

$$\lambda = 0,\ A-\lambda I = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix},\ x_1 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

$$\lambda = 3,\ A-\lambda I = \begin{bmatrix} -2 & 0 & 2 \\ 0 & -4 & -2 \\ 2 & -2 & -3 \end{bmatrix},\ x_2 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

$$\lambda = -3,\ A-\lambda I = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & -2 \\ 2 & -2 & 3 \end{bmatrix},\ x_3 = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$$

$\because\ x_1^T x_2 = 0, x_2^T x_3 = 0, x_1^T x_3 = 0 \quad \therefore\ x_1, x_2, x_3$ are orthogonal.

$$\text{Let } Q = [\,x_1 \quad x_2 \quad x_3\,] = \begin{bmatrix} 2 & 2 & 1 \\ 2 & -1 & -2 \\ -1 & 2 & -2 \end{bmatrix}$$

AQ =[$Ax_1 \quad Ax_2 \quad Ax_3$]

$$=[\,0\times x_1 \quad 3\times Ax_2 \quad -3\times Ax_3\,] = \begin{bmatrix} 2 & 2 & 1 \\ 2 & -1 & -2 \\ -1 & 2 & -2 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix} = Q\Lambda$$

$$Q^{-1}AQ = \Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$Q = \begin{bmatrix} 2 & 2 & 1 \\ 2 & -1 & -2 \\ -1 & 2 & -2 \end{bmatrix}$$

2.(10pts)

(a)

$$B\begin{bmatrix} a \\ b \end{bmatrix} = -\lambda \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow B\begin{bmatrix} y \\ -z \end{bmatrix} = \begin{bmatrix} -Az \\ A^T y \end{bmatrix} = -\lambda \begin{bmatrix} y \\ -z \end{bmatrix}$$

For $-\lambda$, eigenvector = $\begin{bmatrix} y \\ -z \end{bmatrix}$

(b)

$$Az = \lambda y \Rightarrow A^T Az = A^T \lambda y = \lambda(A^T y) = \lambda^2 z$$

3.(30pts)

(a)

False

counter example: $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, $\lambda = 1, 2$, $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b)

True

$$A = Q\Lambda Q^{-1} = Q\Lambda Q^T$$
$$A^T = (Q\Lambda Q^T)^T = Q\Lambda^T Q^T = Q\Lambda Q^T = A$$

(c)

True

$$A = A^T, A^{-1} = (A^T)^{-1} = (A^{-1})^T$$

(d)

False

$$A = A^T \sim B$$
$$B = M^{-1}AM \Rightarrow A = MBM^{-1}$$
$$B^T = (M^{-1}AM)^T = M^T A^T (M^{-1})^T = M^T A (M^T)^{-1}$$

B is not always = B^T

(e)

False

counter example: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\lambda = 2, 0$, $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, S = $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

(f)

True

positive definite $\Rightarrow$ All eigenvalue > 0 $\Rightarrow \lambda_1\lambda_2\lambda_3\cdots\lambda_4 > 0$

(g)

True

$$A = Q\Lambda Q^T,\quad A^{-1} = Q\Lambda^{-1}Q^T,\quad \Lambda^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & \frac{1}{\lambda_2} \end{bmatrix}$$

$\frac{1}{\lambda_1}, \frac{1}{\lambda_2}$ are positive $\Rightarrow$ A^{-1} is positive definite

(h)

True

A is positive definite $u^TAu > 0$, for all u

B is positive definite $u^TBu > 0$, for all u

$u^T(A+B)u = (u^TA+u^TB)u = u^TAu + u^TBu > 0$

(i)

True

All diagonal entries are eigenvalues $\because$ diagonal entries > 0 $\Rightarrow \lambda_i > 0$

$\Rightarrow$ A is positive definite

(j)

True

All projection matrix are singular except I

4.(10pts)

det(A) = -1, det(B) = 0 $\Rightarrow$ A is invertible, B is not invertible

$a_i^T a_j = \begin{cases} 0, i \neq j \\ 1, i = j \end{cases} \Rightarrow$ A is orthogonal, B is not orthogonal

$A^T = A, A^2 \neq A \Rightarrow$ A is not projection

$B^T = B, B^2 = B \Rightarrow$ B is projection

$A^{-1}=A^T \Rightarrow$ A is permutation

B is not permutation(B is not invertible)

$\because$ A is real symmetric $\Rightarrow$ A is diagonalizable $\Rightarrow$ B is diagonalizable

Let det(A- λ I) = 0, λ = 1, 1, -1 $\Rightarrow$ A is not positive definite

det(B) = 0 $\Rightarrow$ B is not positive definite

5.(15pts)

(a)

$x^TAx = (Ax)^Tx = x^TA^Tx = - x^TAx$

$\Rightarrow 2x^TAx = 0$

$\Rightarrow x^TAx = 0$

(b)

Let z be an eigenvector of A

assume $z = a + bi$, $a, b \in R$

$\bar{z}^T Az = \lambda \bar{z}^T z = \lambda \|z\|^2$

$\bar{z}^T Az = (a - bi)^T A(a + bi) = a^T Aa + (a^T Ab)i - (b^T Aa)i + b^T Ab$

$= i(a^T Ab - b^T Aa) \Rightarrow (a^T Ab - b^T Aa)$ is scalar

$\Rightarrow$ The eigenvalue of A are pure imaginary

(c)

Because the eigenvalue of A are pure imaginaty, the eigenvalues are pairs of (bi) and ($-bi$), det(A) = $\lambda_1 \lambda_2 \lambda_3 \cdots \lambda_n \geq 0$

6.(10pts)

$\det(A_1) = c > 0$

$\det(A_2) = c^2 - 1 > 0 \Rightarrow c > 1$ or $c < -1$

$\det(A_3) = c^3 - 3c + 2 = (c-1)^2(c+2) > 0 \Rightarrow c > -2$

$\Rightarrow$ If $c > 1$, it is positive definite

$\det(B_1) = 1 > 0$

$\det(B_2) = d - 9 > 0 \Rightarrow d > 9 \Rightarrow$

$\det(B_3) = d^2 - 9c - 16 > 0 \Rightarrow d > \dfrac{9+\sqrt{145}}{2}$ or $d < \dfrac{9-\sqrt{145}}{2}$

$\Rightarrow$ If $d > \dfrac{9+\sqrt{145}}{2}$, it is positive definite

7.(15pts)

(a)

$$A^TA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow \lambda = 0, 1, 3$$

For $\lambda_1 = 0,\ x_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$; For $\lambda_2 = 1,\ x_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$; For $\lambda_3 = 3,\ x_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$$AA^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow \lambda = 1, 3$$

For $\lambda = 1,\ x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; For $\lambda = 3,\ x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b)

$\sigma_1 = \sqrt{0} = 0, \sigma_2 = \sqrt{1} = 1, \sigma_3 = \sqrt{3}$

(c)

when $x = \dfrac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\max \|Ax\| = \sigma_3 = \sqrt{3}$