Linear Algebra

Problem Set 10 Solution **Spring 2016**

1. (15pts)

(a)

$$T(x) = \left(2\frac{vv^T}{v^Tv} - I\right)x = \begin{bmatrix} -3/5 & 0 & 4/5 \\ 0 & -1 & 0 \\ 4/5 & 0 & 3/5 \end{bmatrix} x$$

$\lambda = 1, -1, -1$

$$\beta = \text{span}\left\{\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}\right\}$$

(b)

$$[T]_\beta = \beta^{-1}T\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(c)

$$T = \begin{bmatrix} -3/5 & 0 & 4/5 \\ 0 & -1 & 0 \\ 4/5 & 0 & 3/5 \end{bmatrix}$$

2.(10pts)

$\lambda = 0, 3, -3$

$$x = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

$$\text{Q} = \begin{bmatrix} -2/3 & 2/3 & -1/3 \\ -2/3 & -1/3 & 2/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix}, \text{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

3.(10pts)

A is a positive definite matrix, all eigenvalues are larger than 0.

$$A = QDQ^T = Q\begin{bmatrix} \sqrt{\lambda_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sqrt{\lambda_n} \end{bmatrix}\begin{bmatrix} \sqrt{\lambda_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sqrt{\lambda_n} \end{bmatrix}Q^T = Q\Sigma\Sigma^T Q^T = BB^T$$

4.(20pts)

(a)True

$\mathrm{rank(A)} = \mathrm{rank}(A^T A) = \mathrm{rank}(A^2)$

(b)True

D is a real diagonal matrix.

$A = QDQ^{-1} = QDQ^T$

$A^T = (QDQ^T)^T = QDQ^T = A$

(c)True

All positive definite matrix's eigenvalues are larger than 0.

$\mathrm{tr(A)} = \mathrm{a + c} > 0, \det(\mathrm{A}) = \mathrm{ac} - b^2 > 0 \rightarrow a > 0, c > 0$

(d)False

正交矩陣必可么正對角化，$A = SDS^{-1} = SDS^*$

S 亦為一么正矩陣，而S與S^*不一定相等。

參考:么正矩陣

(e)False

$A = A^T, A = MBM^{-1}$

$B = M^{-1}AM$

$B^T = M^T A^T (M^{-1})^T = M^T A (M^{-1})^T$

M^{-1} is not always equal to M^T, so $B \neq B^T$.

(f)True

$A^T A$ is a real symmetric matrix → It is diagonalizable.

(g)True

All positive definite matrix's eigenvalues are larger than 0, $\det(A) > 0$.

(h)True

From(g), the eigenvalues of inverse matrix: ${\lambda_n}' = {\lambda_n}^{-1} > 0, n = 1,2,\cdots$.

(i)True

$x^T Ax > 0, x^T Bx > 0$

$x^T (A + B)x = x^T Ax + x^T Bx > 0$

(j)True

$P^2 = P \rightarrow P(P - I) = 0$

$P = 0 \ or \ I$

P is positive definite matrix, so $P = I$.

5.(20pts)

(a)

$A^2 = (-A)(-A) = A^T A^T = (A^2)^T$

$\|Ax\|^2 = x^T A^T Ax = -x^T A^2 x \geq 0 \rightarrow x^T A^2 x \leq 0$

A^2 is symmetric and all eigenvalues are equal to or smaller than zero.

(b)

$x^T Ax = (x^T Ax)^T = x^T A^T x = -x^T Ax = 0$

(c)

$(\bar{A}\bar{x})^T = \left(\bar{\lambda}\bar{x}\right)^T$

$\bar{x}^T \bar{A}^T = \bar{\lambda}\bar{x}^T$

$-\overline{x}^T Ax = -\lambda\overline{x}^T x = \bar{\lambda}\bar{x}^T x$

$-\lambda = \bar{\lambda} \rightarrow \lambda$ is pure imaginary or 0.

(d)

From (c), λ is a complex conjugate pair or zero, so $\det(A)$ is positive or zero.

6.(10pts)

$$\begin{cases} |a| > 0 \\ \begin{vmatrix} a & 1 \\ 1 & a \end{vmatrix} > 0 \\ \det(A) > 0 \end{cases} \rightarrow \begin{cases} a > 0 \\ a^2 - 1 > 0 \\ a^3 - 3a + 2 > 0 \end{cases} \rightarrow \text{a} > 1$$

$$\begin{cases} |1| > 0 \\ \begin{vmatrix} 1 & 3 \\ 3 & b \end{vmatrix} > 0 \\ \det(B) > 0 \end{cases} \rightarrow \begin{cases} b - 9 > 0 \\ b^2 - 9b - 16 > 0 \end{cases} \rightarrow \text{b} > \frac{9 + \sqrt{145}}{2}$$

7.(15pts)

(a)

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$\lambda = 0,1,3$

$\sigma = 0,1,\sqrt{3}$

(b)

$$v = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

(c)

$$x = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\|Ax\| = \left\| \frac{1}{\sqrt{6}}\begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\| = \sqrt{3}$$