Solution to Problem Set 11

1.

$$Let\ \ A=\begin{bmatrix}-2 & 3\\ 2 & -3\end{bmatrix},\frac{d\text{u}}{dt}=\text{Au}$$

$$\det(A-\lambda I)=0$$

$$\rightarrow \lambda_1=0,\lambda_2=-5,x_1=\begin{bmatrix}3\\2\end{bmatrix},x_2=\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\rightarrow \text{u}(t)=c_1e^{0t}\begin{bmatrix}3\\2\end{bmatrix}+c_2e^{-5t}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\text{u}(0)=c_1\begin{bmatrix}3\\2\end{bmatrix}+c_2\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}4\\1\end{bmatrix}$$

$$c_1=1,c_2=1$$

$$\rightarrow \text{u}(t)=\begin{bmatrix}3\\2\end{bmatrix}+e^{-5t}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\therefore \text{u}(\infty)=\begin{bmatrix}3\\2\end{bmatrix}$$

2.

(a)

$$Ax=\lambda x, AAx=A\lambda x=\lambda^2x, A^3x=\lambda^3x,...$$

$$\cos A=I-\frac{A^2}{2!}+\frac{A^4}{4!}-......$$

$$\rightarrow \cos A\cdot x=x(I-\frac{A^2}{2!}+\frac{A^4}{4!}-......)$$

$$=x-\frac{A^2}{2!}x+\frac{A^4}{4!}x-.....$$

$$=x-\frac{\lambda^2}{2!}x+\frac{\lambda^4}{4!}x-.....$$

$$=(1-\frac{\lambda^2}{2!}+\frac{\lambda^4}{4!}-....)x$$

$$\therefore eigenvalue\ \ of\ \ \cos A=1-\lambda^2+\lambda^4-....$$

(b)

$$Ax = \lambda x, Let\ \ x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$Ax_1 = \begin{bmatrix} \pi & \pi \\ \pi & \pi \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\pi \\ 2\pi \end{bmatrix} = 2\pi\begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \lambda_1 = 2\pi$$

$$Ax_2 = \begin{bmatrix} \pi & \pi \\ \pi & \pi \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightarrow \lambda_2 = 0$$

$$\therefore \cos\lambda = 1, \cos A = I$$

3.

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

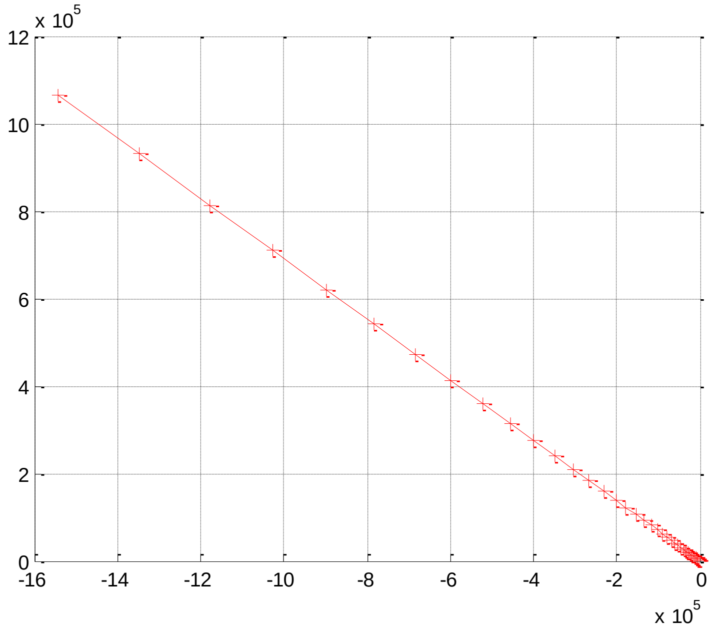
$$\rightarrow \lambda = 2 + i, 2 - i$$

$$(A - (2 - i)I) = x = 0$$

$$\rightarrow x = \begin{bmatrix} 2 \\ -1 + 1\mathrm{i} \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\rightarrow A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}^{-1}$$

4.



5.

(a)

$$B = \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \lambda \begin{bmatrix} y \\ z \end{bmatrix}$$

$$when \;\; eigenvector = \begin{bmatrix} y \\ -z \end{bmatrix}$$

$$\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}\begin{bmatrix} y \\ -z \end{bmatrix} = \lambda^{'} \begin{bmatrix} y \\ -z \end{bmatrix}$$

$$= \begin{bmatrix} \lambda^{'} y \\ -\lambda^{'} z \end{bmatrix} = \begin{bmatrix} Az' \\ -A^T y' \end{bmatrix}$$

$$= -\lambda \begin{bmatrix} y \\ -z \end{bmatrix}$$

(b)

$$A^T Az = A^T(\lambda y) = \lambda(A^T y) = \lambda(\lambda z) = \lambda^2 z$$

$$\rightarrow \lambda^2 \;\; is \;\; an \;\; eigenvalue \;\; of \;\; A^T A$$

(c)

$$When \;\; A = I, B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\det(B - \lambda I) = 0$$

$$\lambda = 1, 1, -1, -1$$

$$x_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, x_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, x_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

6.

(a)

True

Because eigenvalues of $A+I$ *are eigenvalues of* A *plus one.*

They have different eigenvalues so they're not similar.

(b)

True

$\det(A-\lambda I)=0$

$\det(-A-\lambda' I)=\det(A-(-\lambda' I))=0$

$\because \lambda \neq \lambda'$

$\therefore$ *True*

(c)

false

Let $A=\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $B=\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ $P=\begin{bmatrix} 0.5 & 0 \\ 0.5 & 1 \end{bmatrix}$

$A=P^{-1}BP$

B is nonsymmetric

(d)

True

Because A, B *have different rank numbers.*

(e)

True

If A *is similar to* $B \rightarrow A = MBM^{-1}$

$A^2 = (MBM^{-1})(MBM^{-1}) = MB^2M^{-1}$

$\therefore A^2$ *is similar to* B^2

(f)

False

Let $A^2 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, B^2 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

They are similar because they have the same eigenvalue $2, 3$

But one choice of $A = \begin{bmatrix} -\sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix}, B = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{bmatrix}$

They are not similar because they have different eigenvalue.

(g)

True

A= $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ eigenvalue $\lambda_1 = 3$ λ_2=2

eigenvector $X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \rightarrow A = P^{-1}BP$$

故A與B為*similar*

(h)

false

A= $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ B= $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$$B = P^{-1}AP = P^{-1} * 3I * P = 3P^{-1}P = 3I \neq A$$

$\therefore A, B$ *is not similar*.