2013 spring HW2

1.

$$A_{m*n}=\begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} & \cdots & a_{mn} \end{bmatrix} B_{n*m}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & a_{mm} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

$$tr(A*B)=tr(\begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & a_{mm} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix})$$

$$=tr(\begin{bmatrix} a_{11}a_{11}+a_{12}a_{21}+\cdots+a_{1n}a_{n1} & & & \\ & a_{21}a_{12}+a_{22}a_{22}+\cdots+a_{2n}a_{n2} & & \\ & & \ddots & \\ & & & a_{m1}a_{1m}+a_{m2}a_{2m}+\cdots+a_{mn}a_{nm} \end{bmatrix})$$

$$=a_{11}a_{11}+a_{12}a_{21}+\cdots+a_{1n}a_{n1}+a_{21}a_{12}+a_{22}a_{22}+\cdots+a_{2n}a_{n2}+a_{m1}a_{1m}+a_{m2}a_{2m}+\cdots+a_{mn}a_{nm}$$

$$=\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}a_{ji}$$

$$tr(A*B)=tr(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & a_{mm} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} & \cdots & a_{mn} \end{bmatrix})$$

$$=tr(\begin{bmatrix} a_{11}a_{11}+a_{12}a_{21}+\cdots+a_{1m}a_{m1} & & & \\ & a_{21}a_{12}+a_{22}a_{22}+\cdots+a_{2m}a_{m2} & & \\ & & \ddots & \\ & & & a_{n1}a_{1n}+a_{n2}a_{2n}+\cdots+a_{nm}a_{mn} \end{bmatrix})$$

$$=a_{11}a_{11}+a_{12}a_{21}+\cdots+a_{1n}a_{n1}+a_{21}a_{12}+a_{22}a_{22}+\cdots+a_{2n}a_{n2}+a_{m1}a_{1m}+a_{m2}a_{2m}+\cdots+a_{mn}a_{nm}$$

$$=\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ji}a_{ij}$$

$$\because \sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}a_{ji}=\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ji}a_{ij}$$

故得證

2.

(a)

法一

$$\left[\begin{array}{c|c} 0 & b \\ \hline c & 0 \end{array}\right]\left[\begin{array}{c|c} e & f \\ \hline g & h \end{array}\right]=\left[\begin{array}{c|c} \mathrm{I} & 0 \\ \hline 0 & \mathrm{I} \end{array}\right]$$

$bg = I, ce = 0, bh = 0, cf = I$

$g = b^{-1}, e = 0, h = 0, f = c^{-1}$

故可得 $\begin{bmatrix} 0 & C^{-1} \\ B^{-1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1/4 \\ 0 & 0 & 1/3 & 0 \\ 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

法二

觀察法

令 A*B = I

已知B的第一行影響I的第一列,第二行影響I的第二列以此類推

故可知B = $\begin{bmatrix} 0 & 0 & 0 & 1/4 \\ 0 & 0 & 1/3 & 0 \\ 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

法三

高斯消去法

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$$

(b)

法一

$$\left[\begin{array}{c|c} a & 0 \\ \hline 0 & d \end{array}\right]\left[\begin{array}{c|c} e & f \\ \hline g & h \end{array}\right]=\left[\begin{array}{c|c} \mathrm{I} & 0 \\ \hline 0 & \mathrm{I} \end{array}\right]$$

$ae = I, dg = 0, af = 0, dh = I$

$e = a^{-1}, g = 0, f = 0, h = d^{-1}$

故可得 $\begin{bmatrix} a^{-1} & 0 \\ 0 & d^{-1} \end{bmatrix} = \begin{bmatrix} 7 & -2 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$

法二

高斯消去法

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 3 & 7 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 5 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 7 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 5 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & -3 & 2 \end{array}\right]$$

3.

(a)

法一

$A^{-1}+B^{-1}=B^{-1}BA^{-1}+B^{-1}AA^{-1}=B^{-1}(B+A)A^{-1}$取inverse$\Rightarrow A(B+A)^{-1}B$

得證

法二

$(A(A+B)^{-1}B)^{-1}=B^{-1}(A+B)A^{-1}=B^{-1}AA^{-1}+B^{-1}BA^{-1}=B^{-1}+A^{-1}$

得證

(b)

$((I+AB)^{-1}A)^{-1}=A^{-1}(I+AB)=A^{-1}+B\cdots(1)$

$(A(I+BA)^{-1})^{-1}=(I+BA)A^{-1}=A^{-1}+B\cdots(2)$

式子(1)=式子(2)

得證

4.

$A=(I+B)^{-1}(I-B)$

$(I+B)A=I-B$

$B+(I+B)A=I$

$(I+B)+(I+B)A=I+I$

$(I+B)(I+A)=2I$

$(I+B)=2I(I+A)^{-1}$

$$(I+A)^{-1}=\frac{I}{2}(I+B)=\begin{bmatrix} 2 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 2 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 2 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 2 \end{bmatrix}$$

5.

(a)

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}=\begin{bmatrix} 1 & 2 & 0 \\ -3 & 1 & 5 \end{bmatrix}$$

$$\left\{\begin{matrix} -a_1+2c_1=1 \\ b_1+c_1=2 \\ a_1+b_1=0 \end{matrix}\right\}\Rightarrow\left\{\begin{matrix} a_1=-3 \\ b_1=3 \\ c_1=-1 \end{matrix}\right\}$$

$$\left\{\begin{matrix} -a_2+2c_2=-3 \\ b_2+c_2=1 \\ a_2+b_2=5 \end{matrix}\right\}\Rightarrow\left\{\begin{matrix} a_2=5 \\ b_2=0 \\ c_2=1 \end{matrix}\right\}$$

$$X=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}=\begin{bmatrix} -3 & 3 & -1 \\ 5 & 0 & 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & -1 & 0 \\ 6 & -3 & 7 \end{bmatrix}$$

$$\left\{\begin{matrix} a_1 + 3b_1 = -5 \\ \text{-}a_1 = -1 \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} a_1 = 1 \\ b_1 = -2 \end{matrix}\right\}$$

$$\left\{\begin{matrix} a_2 + 3b_2 = 6 \\ \text{-}a_2 = -3 \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} a_2 = 3 \\ b_2 = 1 \end{matrix}\right\}$$

$$X = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

6.

EA=B

$$E\begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & a & 1 \\ 0 & 0 & 1 & 0 & b \end{bmatrix} = \begin{bmatrix} 2 & 6 & 3 & -2 & d \\ 1 & 4 & 3 & c & 19 \\ 0 & -1 & -1 & 1 & e \end{bmatrix}$$

According to 1st~3rd columns of A: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, we get E= $\begin{bmatrix} 2 & 6 & 3 \\ 1 & 4 & 3 \\ 0 & -1 & -1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 6 & 3 \\ 1 & 4 & 3 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & a & 1 \\ 0 & 0 & 1 & 0 & b \end{bmatrix} = \begin{bmatrix} 2 & 6 & 3 & -2 & d \\ 1 & 4 & 3 & c & 19 \\ 0 & -1 & -1 & 1 & e \end{bmatrix}$$

According to 3rd row of E and 4th column of A $\Rightarrow$ a= -1

According to 2nd row of E and 4th column of A $\Rightarrow$ c= -2

According to 2nd row of E and 5th column of A $\Rightarrow$ b=4

According to 1st row of E and 5th column of A $\Rightarrow$ d=24

According to 3rd row of E and 5th column of A $\Rightarrow$ e= -5

7.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix} \xrightarrow{E_{21}(2)} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 4 & 5 \end{bmatrix} \xrightarrow{E_{31}(3)} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & 2 \end{bmatrix} \xrightarrow{E_{32}(2)} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = U$$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}; E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$E = E_{32}E_{31}E_{21}$

$$L = E^{-1} = (E_{32}E_{31}E_{21})^{-1} = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$