Linear Algebra

Problem Set 2 Solution **Spring 2015**

1. (10pts)

$$\begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix} \Rightarrow \begin{bmatrix} a & b & b \\ 0 & a-b & 0 \\ 0 & a-b & a-b \end{bmatrix} \Rightarrow \begin{bmatrix} a & b & b \\ 0 & a-b & 0 \\ 0 & 0 & a-b \end{bmatrix}$$

Pivots=(a , a-b , a-b)

Or $A^{-1}=\begin{bmatrix} \frac{1}{a-b} & 0 & \frac{-b}{a(a-b)} \\ \frac{-1}{a-b} & \frac{1}{a-b} & 0 \\ 0 & \frac{-1}{a-b} & \frac{1}{a-b} \end{bmatrix}$ $\Rightarrow$ A is invertible if $a \neq 0$ and $a \neq b$

2.(15pts)

(a)T $\Rightarrow$ det(A)=0

(b)T $\Rightarrow$ det(A)=0

(c)F $\Rightarrow$ the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not invertible

(d)T $\Rightarrow$ $\det(A^2)=\det(A)\det(A) \neq 0$

(e)F $\Rightarrow$ If $A=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $A+A^T=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

3.(10pts)

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{第一列調到第三列, 第三列調到第四列}]{\text{矩陣的第四列調到第一列, 第二列 x } \frac{1}{2}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$A^{-1}=\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B=\begin{bmatrix} 2 & 9 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}=\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\ ; \quad B^{-1}=\begin{bmatrix} a^{-1} & 0 \\ 0 & d^{-1} \end{bmatrix}=\begin{bmatrix} -4 & 9 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

4.(20pts)

(a)

$$\mathbf{A}(\boldsymbol{A}+\boldsymbol{B})^{-1}\mathbf{B}=(\boldsymbol{B}^{-1}(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{A}^{-1})^{-1}=(\boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{A}^{-1}+\boldsymbol{B}^{-1}\boldsymbol{B}\boldsymbol{A}^{-1})^{-1}$$
$$=(\boldsymbol{B}^{-1}+\boldsymbol{A}^{-1})^{-1}$$
$$\mathbf{B}(\boldsymbol{A}+\boldsymbol{B})^{-1}\mathbf{A}=(\boldsymbol{A}^{-1}(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{B}^{-1})^{-1}=(\boldsymbol{A}^{-1}\boldsymbol{A}\boldsymbol{B}^{-1}+\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{B}^{-1})^{-1}$$
$$=(\boldsymbol{B}^{-1}+\boldsymbol{A}^{-1})^{-1}$$
$$\Rightarrow\ \mathbf{A}(\boldsymbol{A}+\boldsymbol{B})^{-1}\mathbf{B}=\mathbf{B}(\boldsymbol{A}+\boldsymbol{B})^{-1}\mathbf{A}$$

(b) $(\boldsymbol{I}+\boldsymbol{A}^{-1})^{-1}=(\boldsymbol{A}^{-1}\boldsymbol{A}+\boldsymbol{A}^{-1})^{-1}=(\boldsymbol{A}^{-1}(\boldsymbol{A}+\boldsymbol{I}))^{-1}=(\boldsymbol{A}+\boldsymbol{I})^{-1}\mathbf{A}$

$(\boldsymbol{A}^{-1}+\boldsymbol{I})^{-1}=(\boldsymbol{A}^{-1}+\boldsymbol{A}\boldsymbol{A}^{-1})^{-1}=((\boldsymbol{I}+\boldsymbol{A})\boldsymbol{A}^{-1})^{-1}=\boldsymbol{A}(\boldsymbol{A}+\boldsymbol{I})^{-1}$

5.(15pts)

A=(I-B)$(I+B)^{-1}$

A(I+B)=(I-B)

B+A(I+B)=I

(I+B)+A(I+B)=2I

(I+B)(I+A)=2I

(I+B)=2I$(I+A)^{-1}$

$$(I+A)^{-1}=\frac{1}{2}(I+B)=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

6.(10pts)

$$\mathbf{A}\begin{bmatrix} 1 & 5 & 1 \\ 1 & 3 & 3 \\ 0 & 2 & -1 \end{bmatrix}=\begin{bmatrix} 5 & 1 & -2 \\ 3 & 3 & 2 \\ 2 & -1 & -3 \end{bmatrix}$$

$$\Rightarrow A=\begin{bmatrix}5 & 1 & -2\\ 3 & 3 & 2\\ 2 & -1 & -3\end{bmatrix}\begin{bmatrix}1 & 5 & 1\\ 1 & 3 & 3\\ 0 & 2 & -1\end{bmatrix}^{-1}=\begin{bmatrix}24 & -19 & -31\\ 10 & -7 & -13\\ 12.5 & -10.5 & -16\end{bmatrix}$$

$$Ad=\begin{bmatrix}24 & -19 & -31\\ 10 & -7 & -13\\ 12.5 & -10.5 & -16\end{bmatrix}\begin{bmatrix}-2\\ 2\\ -3\end{bmatrix}=\begin{bmatrix}7\\ 5\\ 2\end{bmatrix}$$

7.(10pts)

$$\begin{bmatrix}1 & 2 & 3 & 1 & b\\ 2 & 5 & 3 & a & 0\\ 1 & 0 & 8 & 6 & c\end{bmatrix}=\mathbf{E}^{-1}\begin{bmatrix}1 & 0 & 0 & -2 & 0\\ 0 & 1 & 0 & d & -1\\ 0 & 0 & 1 & 1 & e\end{bmatrix}=\begin{bmatrix}1 & 2 & 3\\ 2 & 5 & 3\\ 1 & 0 & 8\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & -2 & 0\\ 0 & 1 & 0 & d & -1\\ 0 & 0 & 1 & 1 & e\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}1 & 2 & 3\\ 2 & 5 & 3\\ 1 & 0 & 8\end{bmatrix}\begin{bmatrix}-2 & 0\\ d & -1\\ 1 & e\end{bmatrix}=\begin{bmatrix}1 & b\\ a & 0\\ 6 & c\end{bmatrix}$$

-4 + 5d + 3 = a $\Rightarrow$ a = -1

-2 + 3e = b $\Rightarrow$ b = 3

8e = c $\Rightarrow$ c = $\frac{40}{3}$

-2 + 2d + 3 = 1 $\Rightarrow$ d = 0

-5 + 3e = 0 $\Rightarrow$ e = $\frac{5}{3}$

8.(10pts)

$$\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 2 & 3 & 4\\ 1 & 3 & 6 & 10\\ 1 & 4 & 10 & 20\end{bmatrix}\overset{\mathbf{E_{21}}\ \mathbf{E_{31}}\ \mathbf{E_{41}}}{\Longrightarrow}\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 2 & 3\\ 0 & 2 & 5 & 9\\ 0 & 3 & 9 & 19\end{bmatrix}\overset{\mathbf{E_{32}}\ \mathbf{E_{42}}}{\Longrightarrow}\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 2 & 3\\ 0 & 0 & 1 & 3\\ 0 & 0 & 3 & 10\end{bmatrix}\overset{\mathbf{E_{43}}}{\Longrightarrow}\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 2 & 3\\ 0 & 0 & 1 & 3\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\mathbf{E_{43}\cdot E_{42}\cdot E_{32}\cdot E_{41}\cdot E_{31}\cdot E_{21}\cdot A = U}=\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 2 & 3\\ 0 & 0 & 1 & 3\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\mathbf{A} = \mathbf{E_{21}}^{-1}\mathbf{E_{31}}^{-1}\mathbf{E_{41}}^{-1}\mathbf{E_{32}}^{-1}\mathbf{E_{42}}^{-1}\mathbf{E_{43}}^{-1}\mathbf{U} = \mathbf{LU} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$