Linear Algebra

Problem Set 2 Solution **Spring 2016**

1. (20pts)

(a)True

$(A^{-1})^2 \cdot A^2 = A^{-1} \cdot A^{-1} \cdot A \cdot A = I$

$A^2 \cdot (A^{-1})^2 = A \cdot A \cdot A^{-1} \cdot A^{-1} = I$

(b)False

$(A^{-1} + B^{-1}) \cdot (A + B) = I + A^{-1}B + B^{-1}A + I$

$(A + B) \cdot (A^{-1} + B^{-1}) = I + AB^{-1} + BA^{-1} + I$

(c)True

$(B^{-1}A) \cdot (A^{-1}B) = I$

$(A^{-1}B) \cdot (B^{-1}A) = I$

(d)False

$(A^{-1}B^{-1})^T \cdot (AB)^T = (B^{-1})^T(A^{-1})^T B^T A^T$

$(AB)^T \cdot (A^{-1}B^{-1})^T = B^T A^T (B^{-1})^T (A^{-1})^T$

(e)False

$(A + B)^2 = (A + B) \cdot (A + B) = A^2 + AB + BA + B^2$

(f)False

$(A + B)(A - B) = A^2 - AB + BA - B^2$

(g)False

$ABA^{-1} = B \rightarrow ABA^{-1} \cdot A = AB \neq B \cdot A$

(h)True

$(ABA^{-1})^3 = (ABA^{-1})(ABA^{-1})(ABA^{-1}) = AB^3A^{-1}$

(i)True

$(I + A)(I - A) = I^2 - A + A - A^2 = I - A^2$

(j)True

$ABB^{-1}A^{-1} = AIA^{-1} = IAA^{-1} = BB^{-1}A^{-1}A$

2.(20pts)

(a)

Suppose $ABx = 0, B$ is a 2×3 matrix, its pivots ≤ 2.

So, there exists a nonzero vector x such that $Bx = 0$ and let $A(Bx) = 0$.

It proves that AB is not invertible.

(b)

If A is a $n \times n$ singular matrix, there exists a nonzero vector x such that $Ax = 0$.

Let $B = [x \quad x \quad \cdots \quad x]_{n\times n}$ such that $AB = 0$

(c)

$A = \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix} or \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$, a is an arbitrary constant.

(d)

$A^2 = A \rightarrow A^{-1} \cdot A^2 = A^{-1} \cdot A \rightarrow A = I$

3.(15pts)

(a)

$$\begin{aligned}(A+B)A^{-1}(A-B) &= (A+B)(I-A^{-1}B) = A+B-B-BA^{-1}B \\ &= A - BA^{-1}B = (A-B)(I+A^{-1}B) = (A-B)A^{-1}(A+B)\end{aligned}$$

(b)

$$\begin{aligned}AB = A+B \rightarrow\ & AB - A - B = 0 \quad \rightarrow A(B-I)-(B-I) = I \\ & \rightarrow (A-I)(B-I) = I \ \rightarrow (B-I)(A-I) = I \\ & \rightarrow BA - B - A + I = I \ \rightarrow BA = B + A \rightarrow BA = AB\end{aligned}$$

(c)

$(A^{-1}+B^{-1})^{-1} = A - A(A+B)^{-1}A \rightarrow \ (A^{-1}+B^{-1})(A - A(A+B)^{-1}A) = I$

$Let\ (A^{-1}+B^{-1})(A - A(A+B)^{-1}A) = X$

$\rightarrow \ I + B^{-1}A - (A+B)^{-1}A - B^{-1}A(A+B)^{-1}A = X$

$\rightarrow \ B + A - B(A+B)^{-1}A - A(A+B)^{-1}A = BX$

$\rightarrow \ B + A - (B+A)(A+B)^{-1}A = BX$

$\rightarrow \ B + A - A = BX$

$\rightarrow \ X = I$

4.(10pts)

$AB = C \rightarrow B^T A^T = C^T \rightarrow B^T x = C^T$

$$[B^T|C^T] \rightarrow \left[\begin{array}{ccc|cc} -1 & 1 & 3 & 1 & -3 \\ 0 & 1 & 1 & 2 & 1 \\ 1 & 0 & -1 & 0 & 5 \end{array}\right] \rightarrow \left[\begin{array}{ccc|cc} -1 & 1 & 3 & 1 & -3 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 & 2 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|cc} -1 & 1 & 3 & 1 & -3 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|cc} -1 & 1 & 0 & 4 & -6 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|cc} -1 & 0 & 0 & 1 & -6 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|cc} 1 & 0 & 0 & -1 & 6 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right] \rightarrow [I|x]$$

$$A = x^T = \begin{bmatrix} -1 & 3 & -1 \\ 6 & 0 & 1 \end{bmatrix}$$

5.(15pts)

(a)

$PA = LU \rightarrow PAx = LUx = Pb \rightarrow y = Ux \rightarrow Ly = Pb$

$$Ly = Pb \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -3 \\ -3 & 1 & 0 & 0 & 14 \\ 1 & 2 & 1 & 0 & 9 \\ -1 & 8 & -5 & 1 & 33 \end{array}\right] \rightarrow y = \begin{bmatrix} -3 \\ 5 \\ 2 \\ 0 \end{bmatrix}$$

$$Ux = y \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & -1 & 4 & -3 \\ 0 & 1 & 3 & 7 & 5 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \rightarrow x = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$$

(b)

$PA = LU \rightarrow A = P^T LU$

Let x be the first row of $A^{-1} \rightarrow xA = xP^T LU = [1 \quad 0 \quad 0 \quad 0] = \mathrm{c}$

$U^T L^T P x^T = c^T \rightarrow L^T P x^T = y \rightarrow U^T \mathrm{y} = c^T$

$$U^T \mathrm{y} = c^T \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 0 \\ -1 & 3 & 1 & 0 & 0 \\ 4 & 7 & 2 & 1 & 0 \end{array}\right] \rightarrow y = \begin{bmatrix} 1 \\ -2 \\ 7 \\ -4 \end{bmatrix}$$

$$L^T P x^T = y \rightarrow \left[\begin{array}{cccc|c} -3 & 1 & -1 & 1 & 1 \\ 1 & 0 & 8 & 2 & -2 \\ 0 & 0 & -5 & 1 & 7 \\ 0 & 0 & 1 & 0 & -4 \end{array}\right] \rightarrow x^T = \begin{bmatrix} 56 \\ 178 \\ -4 \\ -13 \end{bmatrix}$$

6.(20pts)

(a)

$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$，$B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}$，$0 \le a_{ij} \le p$，$1 \le (\mathrm{i,j}) \le \mathrm{n}$

$(b_{11} + b_{21} + \cdots b_{n1})$、$(b_{12} + b_{22} + \cdots b_{n2})$、 … 、$(b_{1n} + b_{2n} + \cdots b_{nn}) \le q$

$0 \le b_{ij}$，$1 \le (\mathrm{i,j}) \le \mathrm{n}$

Let $C = AB$

$c_{11} = a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1} \le p(b_{11} + b_{21} + \cdots + b_{n1}) \le pq$

$\vdots$

$c_{nn} = a_{n1}b_{1n} + a_{n2}b_{2n} + \cdots + a_{nn}b_{nn} \le p(b_{1n} + b_{2n} + \cdots + b_{nn}) \le pq$

(b)

$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$

$(a_{11} + a_{21} + \cdots + a_{n1})$、$(a_{12} + a_{22} + \cdots + a_{n2})$、 … 、$(a_{1n} + a_{2n} + \cdots + a_{nn}) < 1$

$a_1 = a_{11} + a_{21} + \cdots + a_{n1}$， a_2 ， ⋯ ，$a_n = a_{1n} + a_{2n} + \cdots a_{nn}$

$\max_{1 \le i \le n} a_i = s$

考慮一最大可能性：某列之元素皆為 s

Let $B = A^2$,

$b_{11} = sa_{11} + sa_{21} + \cdots + sa_{n1} = s(a_{11} + a_{21} + \cdots + a_{n1}) = sa_1 \le s^2$,

all entries of $A^2 \le s^2$.

Then, $A^m = A^{m-1} \cdot A \rightarrow s^{m-1}a_{11} + s^{m-1}a_{21} + \cdots + s^{m-1}a_{n1} = s^{m-1}a_1 \le s^m$

$\therefore$ all entries of $A^m \le s^m$

(c)

Because $s < 1$, so $\lim_{m \to \infty} s^m = 0$, and all entries of $A^m \le s^m$.

That is, $\lim_{m \to \infty} A^m = 0$.