Solution to Problem Set 3

1.

let $A = E - I$, $A^{-1} = (E-I)^{-1}$

assume $(E-I)^{-1}=(xE-I)$ x is a scalar

$(xE-I)(E-I)=I$

$xE^2-xE-E+I=I$ $E^2=nE$

$nxE-xE-E=0$

$nx-x-1=0$

$x=\frac{1}{n-1}$

$A^{-1} = (E-I)^{-1}=(xE-I)=\frac{1}{n-1}E - I$

2.

The solution is based on 每周問題 March 8, 2010
http://ccjou.files.wordpress.com/2010/03/powsol-march-8-10.pdf

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 3 & 6 & 6 & 8 & 2 \\ 1 & 3 & 4 & 6 & 3 \\ 2 & 5 & 7 & 9 & 4 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ \underline{3} & 0 & -3 & -4 & 2 \\ \underline{1} & 1 & 1 & 2 & 3 \\ \underline{2} & 1 & 1 & 1 & 4 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ \underline{1} & 1 & 1 & 2 & 3 \\ \underline{3} & 0 & -3 & -4 & 2 \\ \underline{2} & 1 & 1 & 1 & 4 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ \underline{1} & 1 & 1 & 2 & 3 \\ \underline{3} & 0 & -3 & -4 & 2 \\ \underline{2} & \underline{1} & 0 & -1 & 4 \end{array}\right]$$

$$\therefore P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} U = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -3 & -4 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

3. A = B + C

$A^T = B^T + C^T = B - C$

∴ $B = \frac{1}{2}(A+A^T)$ $C = \frac{1}{2}(A-A^T)$

$$B = \begin{bmatrix} 1 & 2.5 & 2 & 3 \\ 2.5 & 6 & 4.5 & 6.5 \\ 2 & 4.5 & 4 & 6.5 \\ 3 & 6.5 & 6.5 & 9 \end{bmatrix} \quad C = \begin{bmatrix} 0 & -0.5 & 1 & 1 \\ 0.5 & 0 & 1.5 & 1.5 \\ -1 & -1.5 & 0 & -0.5 \\ -1 & -1.5 & 0.5 & 0 \end{bmatrix}$$

4.

(a)F ex: $A=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $A^2=0$

(b)T (前提:A 為實數矩陣)

但若 A 為虛數矩陣, 則 $A^TA \neq 0$ ex:$A=\begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$

(c)T A is symmetric，$A^TA = AA$ is also symmetric

∴$A^k=AA\cdots A$ is also symmetric.

(d)F ∵AB 不一定等於 BA

(e)T 呈(c) A^2,B^2 is symmetric，令

$$A^2 = \begin{bmatrix} a_{11} \cdots\cdots a_{m} \\ \vdots \\ a_{m1} \cdots\cdots a_{m} \end{bmatrix} B^2 = \begin{bmatrix} b_{11} \cdots\cdots b_{m} \\ \vdots \\ b_{m1} \cdots\cdots b_{m} \end{bmatrix}$$, $a_{ij}=a_{ji}$ $b_{ij}=b_{ji}$

$$A^2 - B^2 = \begin{bmatrix} (a_{11}-b_{11})\,(a_{12}-b_{12}) \cdots\cdots (a_{1n}-b_{1n}) \\ (a_{12}-b_{12})(a_{22}-b_{22}) \cdots\cdots (a_{2n}-b_{2n}) \\ \vdots \quad \ddots \\ \vdots \quad\quad \ddots \\ \vdots \quad\quad\quad \ddots \\ (a_{1n}-b_{1n})(a_{2n}-b_{2n}) \cdots\cdots (a_{mn}-b_{mn}) \end{bmatrix}$$, still symmetric

5.

There are n! permutation matirces of order n.Eventually two powers of P must be the same:
If $P^r = P^s$ then $P^{r-s} = I$. Certainly r-s $\leq$ n!

$$P=\begin{bmatrix} P_2 & \\ & P_3 \end{bmatrix}$$, $$P_2=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, P_3=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

6.

S is symmetric ∴S=LDU=LDL^T

$$S=\begin{bmatrix} \mathrm{Im} & A \\ A^T & 0 \end{bmatrix}=LU=\begin{bmatrix} I_m & 0 \\ M & I_n \end{bmatrix}\begin{bmatrix} I_m & A \\ 0 & N \end{bmatrix}$$

M,N:未知

$M=A^T$

MA+N=0 $N=-A^TA$

$$\therefore S=\begin{bmatrix} I_m & 0 \\ M & I_n \end{bmatrix}\begin{bmatrix} I_m & A \\ 0 & N \end{bmatrix}=\begin{bmatrix} I_m & 0 \\ A^T & I_n \end{bmatrix}\begin{bmatrix} I_m & A \\ 0 & -A^TA \end{bmatrix}$$

$$=\begin{bmatrix} I_m & 0 \\ A^T & I_n \end{bmatrix}\begin{bmatrix} I_m & 0 \\ 0 & -A^TA \end{bmatrix}\begin{bmatrix} I_m & A \\ 0 & I_n \end{bmatrix}=LDL^T$$

$$S^{-1} = (L^T)^{-1} D^{-1} L^{-1}$$

$$= \begin{bmatrix} I_m & \mathrm{A} \\ 0 & I_n \end{bmatrix}^{-1} \begin{bmatrix} I_m & 0 \\ 0 & -A^T A \end{bmatrix}^{-1} \begin{bmatrix} I_m & 0 \\ A^T & I_n \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} I_m & -\mathrm{A} \\ 0 & I_n \end{bmatrix} \begin{bmatrix} I_m & 0 \\ 0 & (-A^T A)^{-1} \end{bmatrix} \begin{bmatrix} I_m & 0 \\ -A^T & I_n \end{bmatrix}$$

$$= \begin{bmatrix} I_m & (A^T)^{-1} \\ 0 & (-A^T A)^{-1} \end{bmatrix} \begin{bmatrix} I_m & 0 \\ -A^T & I_n \end{bmatrix}$$

$$= \begin{bmatrix} 0 & (A^T)^{-1} \\ A^{-1} & (-A^T A)^{-1} \end{bmatrix}$$