Linear Algebra

Problem Set 3 Solution **Spring 2015**

1.(15pts)

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} = LU = \begin{bmatrix} I & 0 \\ B^T A^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ 0 & C - B^T A^{-1} B \end{bmatrix} = LDU = \begin{bmatrix} I & 0 \\ B^T A^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & C - B^T A^{-1} B \end{bmatrix} \begin{bmatrix} I & A^{-1} B \\ 0 & I \end{bmatrix}$$

$$= LDL^T$$

where $\begin{bmatrix} I & 0 \\ B^T A^{-1} & I \end{bmatrix}^T = \begin{bmatrix} I & A^{-1} B \\ 0 & I \end{bmatrix}$ (A is symmetric).

2.(10pts)

$A = B + C$

$A^T = B^T + C^T = B - C$

$B = \dfrac{A + A^T}{2}$

$C = \dfrac{A - A^T}{2}$

3.(15pts)

(a) Yes.

(b) NO. Let $V=\{(b_1,b_2,b_3) \mid b_1 = 2\}$, $(0,0,0) \notin V$

(c) NO. Let $V=\{(b_1,b_2,b_3) \mid b_1 b_2 b_3 = 0\}$, $(0,1,1) \in V$, $(1,0,0) \in V$, but $(0,1,1)+(1,0,0) =(1,1,1) \notin V$.

(d) YES.

(e) NO. Let $V=\{(b_1,b_2,b_3) \mid b_1+ b_2+ b_3 = 1\}$ $(0,0,0) \notin V$

(f) NO. Let $V=\{(b_1,b_2,b_3) \mid b_1 \le b_2 \le b_3\}$, $(1,2,3) \in V$, but $-1(1,2,3) = (-1,-2,-3) \notin V$

4.(10pts)

(a)True. $\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix} \in M$, $\begin{bmatrix}5 & 4\\ 4 & 7\end{bmatrix} \in M$, $\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}+\begin{bmatrix}5 & 4\\ 4 & 7\end{bmatrix}=\begin{bmatrix}6 & 6\\ 6 & 10\end{bmatrix} \in M$

(b)True. $\begin{bmatrix}0 & -2\\ 2 & 0\end{bmatrix} \in M$, $\begin{bmatrix}0 & 4\\ -4 & 0\end{bmatrix} \in M$, $\begin{bmatrix}0 & -2\\ 2 & 0\end{bmatrix}+\begin{bmatrix}0 & 4\\ -4 & 0\end{bmatrix}=\begin{bmatrix}0 & 2\\ -2 & 0\end{bmatrix} \in M$

(c)False. $\begin{bmatrix}1 & 2\\ 1 & 0\end{bmatrix} \in M$, $\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} \in M$, $\begin{bmatrix}1 & 2\\ 1 & 0\end{bmatrix}+\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}=\begin{bmatrix}2 & 2\\ 2 & 1\end{bmatrix} \notin M$

(d)False. $\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} \in M$, $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \in M$, $\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}2 & 1\\ 0 & 1\end{bmatrix} \notin M$

5.(20pts)

(a)Flase. Let A= $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$, $\begin{bmatrix}1\\ 1\end{bmatrix} \notin C(A)$, $\begin{bmatrix}0\\ -1\end{bmatrix} \notin C(A)$, but, $\begin{bmatrix}1\\ 1\end{bmatrix}+\begin{bmatrix}0\\ -1\end{bmatrix}=\begin{bmatrix}1\\ 0\end{bmatrix} \in C(A)$

(b)True.

(c)False. Let A= $\begin{bmatrix}1 & 2\\ 1 & 0\end{bmatrix}$, $A^2 = \begin{bmatrix}3 & 1\\ 2 & 2\end{bmatrix}$ C(A)=span{ $\begin{bmatrix}1\\ 1\end{bmatrix}$, $\begin{bmatrix}2\\ 0\end{bmatrix}$ }, C(A^2)=span{ $\begin{bmatrix}3\\ 2\end{bmatrix}$, $\begin{bmatrix}1\\ 2\end{bmatrix}$ }

(d)True.

(e)True.

(f)False. Let A= $\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}$, $A^T = \begin{bmatrix}1 & 2\\ 1 & 2\end{bmatrix}$, N(A) = span{ $\begin{bmatrix}-1\\ 1\end{bmatrix}$ }, N(A^T) = span{ $\begin{bmatrix}-2\\ 1\end{bmatrix}$ },

(g)False. Let A= $\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}$, $A^T = \begin{bmatrix}1 & 2\\ 1 & 2\end{bmatrix}$, C(A)=span{ $\begin{bmatrix}1\\ 2\end{bmatrix}$ }, C(A^T)=span{ $\begin{bmatrix}1\\ 1\end{bmatrix}$ }

(h)False. Let A= $\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}$, $A^T = \begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}$, free variables are not the same

6.(10pts)

Let A= $\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$, N(A) = span{ $\begin{bmatrix}1\\ 0\end{bmatrix}$ } = C(A)

7.(10pts)

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

8.(10pts)

$$A= \begin{bmatrix} 2 & 4 & 6 & 2 \\ 2 & 5 & 7 & 3 \\ 2 & 3 & 5 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 4 & 6 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & -2 & -2 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 4 & 6 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ let } x_3 = c_1, \quad x_4 = c_2$$

$x_1 + c_1 - c_2 = 0 \Rightarrow x_1 = c_2 - c_1$

$x_2 + c_1 + c_2 = 0 \Rightarrow x_2 = -(c_1 + c_2)$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} c_2 - c_1 \\ -(c_2 + c_1) \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} c_1 + \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} c_2$$

$$N(A) = \text{span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}, \text{ nullspace matrix } N = \begin{bmatrix} -1 & 1 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$