Linear Algebra

Problem Set 3 Solution **Spring 2016**

1. (10pts)

$$S^{-1}AS = \begin{bmatrix}1 & 0\\0 & D\end{bmatrix}^{-1}\begin{bmatrix}a & b^T\\0 & C\end{bmatrix}\begin{bmatrix}1 & 0\\0 & D\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & D^{-1}\end{bmatrix}\begin{bmatrix}a & b^T\\0 & C\end{bmatrix}\begin{bmatrix}1 & 0\\0 & D\end{bmatrix}$$

$$= \begin{bmatrix}a & b^T\\0 & D^{-1}C\end{bmatrix}\begin{bmatrix}1 & 0\\0 & D\end{bmatrix} = \begin{bmatrix}a & b^TD\\0 & D^{-1}CD\end{bmatrix}$$

2.(10pts)

$$A = \begin{bmatrix}I_n & b\\c^T & 1\end{bmatrix} = \begin{bmatrix}I_n & 0\\c^T & 1\end{bmatrix}\begin{bmatrix}I_n & b\\0 & 1-c^Tb\end{bmatrix} = \begin{bmatrix}I_n & 0\\c^T & 1\end{bmatrix}\begin{bmatrix}I_n & 0\\0 & 1-c^Tb\end{bmatrix}\begin{bmatrix}I_n & b\\0 & 1\end{bmatrix} = LDU$$

If there exists A^{-1}, then $A^{-1} = U^{-1}D^{-1}L^{-1}$, and $D^{-1} = \begin{bmatrix}(I_n)^{-1} & 0\\0 & (1-c^Tb)^{-1}\end{bmatrix}$

That is, $1 - c^Tb \neq 0 \rightarrow \boldsymbol{c^Tb \neq 1}$.

$$A^{-1} = \begin{bmatrix}I_n & -b\\0 & 1\end{bmatrix}\begin{bmatrix}I_n & 0\\0 & (1-c^Tb)^{-1}\end{bmatrix}\begin{bmatrix}I_n & 0\\-c^T & 1\end{bmatrix} = \begin{bmatrix}I_n & -b*(1-c^Tb)^{-1}\\0 & (1-c^Tb)^{-1}\end{bmatrix}\begin{bmatrix}I_n & 0\\-c^T & 1\end{bmatrix}$$

$$= \begin{bmatrix}I_n+b*(1-c^Tb)^{-1}*c^T & -b*(1-c^Tb)^{-1}\\(1-c^Tb)^{-1}*c^T & (1-c^Tb)^{-1}\end{bmatrix}, c^Tb \neq 1$$

3.(10pts)

(a) $(0,0,0) \notin S$

(b) $(1,0,0) \in S,\ (0,1,1) \in S,\ \text{but}(1,0,0) + (0,1,1) = (1,1,1) \notin S.$

(c) $(-1,1,1) \in S,\ (-1,1,2) \in S,\ \text{but}(-1,1,1) + (-1)(-1,1,2) = (0,0,-1) \notin S.$

(d) It is a subspace of $\mathbb{R}^3$.

(e) It is a subspace of $\mathbb{R}^3$.

4.(10pts)

Let $\mathrm{V} = (x_1, x_2, \cdots, x_m)$ be a list vectors in $\mathbb{R}^n$.
That is, $\mathrm{span(V)} = c_1x_1 + c_2x_2 + \cdots + c_mx_m$ for some $c_1, c_2, \cdots, c_m \in \mathcal{R}$.

$\mathbf{0} \in \mathbf{span(V)}$ when $c_1, c_2, \cdots, c_m = 0$

Suppose $a, b \in \mathrm{span(V)}$, where $a = c_1x_1 + c_2x_2 + \cdots + c_mx_m$,
$b = d_1x_1 + d_2x_2 + \cdots + d_mx_m$ for $c_1, c_2, \cdots, c_m,\ d_1, d_2, \cdots, d_m \in \mathcal{R}$
and scalar $\lambda \in \mathcal{R}$.

Where $\boldsymbol{a} + \boldsymbol{\lambda b} = (c_1 + \lambda d_1)x_1 + (c_2 + \lambda d_2)x_2 + \cdots + (c_m + \lambda d_m)x_m$
$\in \mathbf{span(V)}$

Hence, $\mathrm{span(V)}$is a subspace of $\mathbb{R}^n$.

5.(15pts)

(a)

$$\text{Let } A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, \qquad A\begin{bmatrix} 5 \\ 3 \\ -8 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{2}{5} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{8} \end{bmatrix}$$

(b)

$y \in col(A)\ and\ rank(A) = 1$

$$\text{Let } A = \begin{bmatrix} 2a & 2b & 2c \\ 0 & 0 & 0 \\ a & b & c \end{bmatrix}, 5a + 3b - 8c = 1 \rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = c_1\begin{bmatrix} 3 \\ -5 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 8 \\ 0 \\ 5 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$$

(c)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}, \begin{cases} 5a_{11} + 3a_{12} - 8a_{13} = 2 \\ 3a_{22} - 8a_{23} = 0 \\ -8a_{33} = 1 \end{cases} \qquad a_{11}, a_{22}, a_{33} \neq 0$$

$$\begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \end{bmatrix} = c_1\begin{bmatrix} 3 \\ -5 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 8 \\ 0 \\ 5 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} a_{22} \\ a_{23} \end{bmatrix} = c_1\begin{bmatrix} 8 \\ 3 \end{bmatrix}, \quad a_{33} = -\frac{1}{8}$$

(d)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \begin{cases} 5a_{11} + 3a_{12} - 8a_{13} = 2 \\ 5a_{21} + 3a_{22} - 8a_{23} = 0 \\ 5a_{31} + 3a_{32} - 8a_{33} = 1 \end{cases}$$

$$\begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ -5 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 8 \\ 0 \\ 5 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} a_{21} \\ a_{22} \\ a_{23} \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ -5 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 8 \\ 0 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} a_{31} \\ a_{32} \\ a_{33} \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ -5 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 8 \\ 0 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}, \text{all entries are nonzero.}$$

6.(15pts)

(a)

$$A = \begin{bmatrix} \alpha & \beta \\ 3\alpha & 3\beta \end{bmatrix}, \begin{bmatrix} \alpha & \beta \\ 3\alpha & 3\beta \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\alpha + 3\beta = 0, \alpha, \beta$ are arbitrary nonzero constants.

(b)

$C(A)$ is a plane spanned by two vectors, which are orthogonal to $(1,2,3)$.

$x + 2y + 3z = 0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

$$C(A) = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}, \ A = \begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(c)

$$A = [a \quad b \quad c], [a \quad b \quad c] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\because x + 2y + 3z = 0$

$\therefore A = [1 \quad 2 \quad 3]$

(d)

$\mathrm{N}(A) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}\right\}$, and A is an m by 3 matrix.

By $\text{rank} - \text{nullity theorem}, \dim N(A) + \dim C(A) = 3, so \dim C(A) = 2.$

The minimal value of m is 2.

Suppose $A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and x_3 is a free variable.

The nullspace matrix is $N = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} -F \\ 1 \end{bmatrix}.$

Thus, $\mathrm{rref}(A) = [I_2 \quad F] = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{bmatrix}$

7.(10pts)

$N(B) = C(A) \rightarrow BA = 0 \rightarrow A^T B^T = 0$

$$A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \\ 6 & 4 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & -2 & -4 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$B^T = \begin{bmatrix} 1 & 2 \\ -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = X\omega, X \text{ is } A^T\text{'s nullspace matrix.}$

$B = X^T = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{bmatrix}$

8.(20pts)

(a)

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & a \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, a is an arbitrary constant.

(b)

$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \end{bmatrix}, \begin{bmatrix} 1 & a & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & a & b \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & a \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, a and b are arbitrary constants.

(c)

藉由 Gauss-Jordan elimination，可以將一 m 列 n 行的矩陣 A 化簡為簡約列梯矩陣 R (e.g. : $R = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 5 \end{bmatrix}$)，而 Gauss-Jordan elimination 則是從列運算而來。相反地，可將簡約列梯矩陣 R 經由適當的列運算轉換為原矩陣 A。

(d)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(e)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$