Solutions to Problem Set 4

1. (a)(f)

(b)The vector does not pass through the origin.

(c)Let $S=\{(x_1,x_2,x_3,x_4)|x_1x_3=0\}$

$v_1 = (1,0,0,0)\in S$

$v_2 = (0,0,1,0)\in S$

$v_1 + v_2 \notin =(1,0,1,0) \notin S$

(d)Let $S=\{(x_1,x_2,x_3,x_4)|x_1\leq x_2\}$

$v_1=(1,2,0,0)\in S$

let c = -1 $cv_1=(-1,-2,0,0)\notin S$

(e)The vector does not pass through the origin.

2.

(a)T

(b)F

Let $s_1=(1,1,1)\in S\cup T$ $t_1=(1,2,3)\in S\cup T$

$s_1+ t_1=(2,3,4) \notin S\cup T$

(c)T

3.

(a)T

N(2A)=N(A)

(b)F

$$ex: A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} \quad A^2=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$But\ C(A)\neq\{0\}$

(c)F

$$ex: A=\begin{bmatrix}1 & 2\\ 3 & 8\end{bmatrix} \quad A^2=\begin{bmatrix}7 & 18\\ 27 & 70\end{bmatrix}$$

$C(A)\neq C(A^2)$

(d)F

$ex: A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} \quad A^2=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

$But\ N(A)\neq N(A^2)$

4.

(a)The nullspace of B $=\begin{bmatrix}A & A\end{bmatrix}$ contains all vector x$=\begin{bmatrix}y\\ -y\end{bmatrix}$ for y in R^4.

(b) if Cx = $\begin{bmatrix}A\\ B\end{bmatrix}$x = 0 ,then Ax = 0 and Bx = 0

So N(C) = N(A)∩N(B)

5.

(a)solution **x** = x_h+x_p = span$\left\{\begin{bmatrix}1\\0\\5\\2\end{bmatrix}\begin{bmatrix}6\\0\\1\\2\end{bmatrix}\right\}$

→So we can know that $x_p=0$

→b=0

$$\text{span}\left\{\begin{bmatrix}1\\0\\5\\2\end{bmatrix}\begin{bmatrix}6\\0\\1\\2\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}-\frac{5}{4}\\0\\1\\0\end{bmatrix}\begin{bmatrix}\frac{29}{8}\\0\\0\\1\end{bmatrix}\right\}$$

→We can get an A = $\begin{bmatrix}c_1(1 & 0 & \frac{5}{4} & \frac{-29}{8})\\ c_2(0 & 1 & 0 & 0\ )\\ 0 & 0 & 0 & 0\end{bmatrix}$, c_1 c_2∈ R

(b) R=$\begin{bmatrix}1 & 0 & \frac{5}{4} & -\frac{29}{8}\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$

(c) $RX = \begin{bmatrix} 1 & 0 & \frac{5}{4} & -\frac{29}{8} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$

$X = \text{span}\left\{ \begin{bmatrix} -\frac{5}{4} \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} \frac{29}{8} \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Matrix $N = \begin{bmatrix} \frac{-5}{4} & \frac{29}{8} \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

(d) Let $S = C(N) = \text{span}\left\{ \begin{bmatrix} -\frac{5}{4} \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} \frac{29}{8} \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} = c_1 \begin{bmatrix} \frac{-5}{4} \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \frac{29}{8} \\ 0 \\ 0 \\ 1 \end{bmatrix} = \text{span}\{m,n\}$

$T = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 5 \\ 2 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\} = c_3 \begin{bmatrix} 1 \\ 0 \\ 5 \\ 2 \end{bmatrix} + c_4 \begin{bmatrix} 6 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \text{span}\{u,v\}$ $\qquad c_1, c_2, c_3, c_4 \in R$

Let $t \in T = au + bv \qquad a,b \in R$

$= (\frac{1}{4} c_3 u - \frac{1}{4} c_4 v) + (\frac{-1}{8} c_3 u + \frac{5}{8} c_4 v) = c_1 \begin{bmatrix} \frac{-5}{4} \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \frac{29}{8} \\ 0 \\ 0 \\ 1 \end{bmatrix} \in S$

Let $s \in S = cm + dn \qquad c,d \in R$

$= (5m + 2n) + (m + 2n) = c_3 \begin{bmatrix} 1 \\ 0 \\ 5 \\ 2 \end{bmatrix} + c_4 \begin{bmatrix} 6 \\ 0 \\ 1 \\ 2 \end{bmatrix} \in T$

thus $S \subseteq T$, $T \subseteq S$, and therefore $S=T$.

6.

(a) if r = m

→R = [I F]

→We can choose a matrix B to let the linear combination of the columns in R form an identity matrix I.

→So it is obvious to show that $B=\begin{bmatrix} I \\ 0 \end{bmatrix}$, I is m by m and B is n by m.

(b) if r = n

→$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$, $R^T = [I\ 0]$

→$R^T C^T = I$. We can also choose a matrix C^T to let the linear combination of the columns in R^T form an identity matrix I.

→So it is obvious to show that C=[I D], C is n by m.

(c)

$$R^T = \begin{bmatrix} I & 0 \\ F^T & 0 \end{bmatrix}$$

→R^T is n by m, I is r by r and F^T is (n-r) by r.

$$\longrightarrow \begin{bmatrix} I & 0 \\ F^T & 0 \end{bmatrix} \xrightarrow{R_{12}(-F^T)} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

→The reduced row echelon form of $R^T = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$

(d)

$$R^T R = \begin{bmatrix} I & F \\ F^T & F^T F \end{bmatrix}$$

→$R^T R$ is n by n, I is r by r, F^T is (n-r) by r and $F^T F$ is (n-r) by (n-r)

$$\rightarrow R^T R = \begin{bmatrix} I & F \\ F^T & F^T F \end{bmatrix} \xrightarrow{R_{12}(-F^T)} \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

→The reduced row echelon form of $R^T R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

(e)

$$RR^T = \begin{bmatrix} I_r & F \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_r & 0 \\ F^T & 0 \end{bmatrix}$$

$$= \begin{bmatrix} I_r + FF^T & 0 \\ 0 & 0 \end{bmatrix}$$

→But we do not know F ,so we cannot invert $I_r + FF^T$ to I_r .

→We cannot tell the reduced echelon form of RR^T .