2013 spring HW4

1.

(a) It is impossible.

A is a $3\times n$ matrix $\quad \dim C(A)\ge 2 \quad \dim N(A)\ge 2 \quad \text{rank}(A)\le 3$

By rank-nullity: $3\ge \text{rank}(A)=\dim C(A)+\dim N(A)\ge 2+2=4$ 矛盾

(b) $A=\begin{bmatrix}1&1&1\\1&0&0\\1&0&0\end{bmatrix}$ or $\begin{bmatrix}1&1&1\\0&1&1\\0&1&1\end{bmatrix}$ which C(A) and $C(A^T)$contains $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$

(c) $A=\begin{bmatrix}1&-1\\1&-1\\1&-1\end{bmatrix}$ which C(A) has basis $\begin{bmatrix}1\\1\\1\end{bmatrix}$，$R=\begin{bmatrix}1&-1\\0&0\\0&0\end{bmatrix}$，$N=\begin{bmatrix}1\\1\end{bmatrix}$，N(A) has basis $\begin{bmatrix}1\\1\end{bmatrix}$

(d) It is impossible.

A is a $m\times n$ matrix which $n=3$，$\text{rank}(A)\le 3$

$\dim C(A)=\dim C(A^T)=1 \qquad \dim N(A)=1$

Suppose $\text{rank}(A)=3$

By rank-nullity: $3=\text{rank}(A)\ne \dim C(A)+\dim N(A)=1+1=2$

(e)

$\because$ column space = row space $\therefore$ A must be square $(m=n)$

$\Rightarrow \dim N(A)=n-r=m-r=\dim N(A^T)$

For example, $A=\begin{bmatrix}1&1\\1&1\end{bmatrix}=A^T, R=\begin{bmatrix}1&1\\0&0\end{bmatrix}, N(A)=\text{span}\left\{\begin{bmatrix}-1\\1\end{bmatrix}\right\}=N(A^T)$

2.

(a)

$$A=\begin{bmatrix}1&4&6&2\\1&5&7&3\\1&3&5&1\end{bmatrix}\xrightarrow{E_{21}}\begin{bmatrix}1&4&6&2\\0&1&1&1\\1&3&5&1\end{bmatrix}\xrightarrow{E_{31}}\begin{bmatrix}1&4&6&2\\0&1&1&1\\0&-1&-1&-1\end{bmatrix}$$

$$\xrightarrow{E_{32}}\begin{bmatrix}1&4&6&2\\0&1&1&1\\0&0&0&0\end{bmatrix}\xrightarrow{E_{12}}\begin{bmatrix}1&0&2&-2\\0&1&1&1\\0&0&0&0\end{bmatrix}=R$$

(b)

$$N=\begin{bmatrix}-2&2\\-1&-1\\1&0\\0&1\end{bmatrix}$$

(c)

$$Ax_h = 0 \Rightarrow x_h = a\begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

(d)

If $b \in C(A)$,then Ax=b is solvable.

$$C(A)=\text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}\right\} \Rightarrow b = \alpha\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \beta\begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}$$

(e)

$Ax_p = b \Rightarrow \text{Set free variable} = 0$

$$x_p = \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 6 & 2 \\ 1 & 5 & 7 & 3 \\ 1 & 3 & 5 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 11 \\ 7 \end{bmatrix} \Rightarrow x_p = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$x = x_h + x_p = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + a\begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

3.Matrix is $m \times n$, rank=r

(1) $\begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix} \Rightarrow r = 1, m = 3, n = 2$ (2) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \Rightarrow r = 1, m = 2, n = 3$ (3) $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \Rightarrow r = 2, m = 3, n = 2$

(4) $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \Rightarrow r = 2, m = 2, n = 3$ (5) $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \Rightarrow r = 2, m = 2, n = 2$

(a) (4)(5)

At least one solution $\Leftrightarrow$ full row rank $\Leftrightarrow r = m$

(b) (3)(5)

At most one solution $\Leftrightarrow$ full column rank $\Leftrightarrow r = n$

(c) (5)

Exactly one solution $\Leftrightarrow$ full rank $\Leftrightarrow r = n = m$

(d) (4)

∞ solutions $\Leftrightarrow r < n$，$r = m$

(e) (1)(2)(3)

no solution for some b $\Leftrightarrow r < m$

4.

(a) $\text{rank}(A) = n$ ($\because$ columns of A is linear independent)

(b)

$\because$ columns of A is linear independent $\therefore$ Reduced row echelon form of A is $\begin{bmatrix} I_n \\ \hline 0 \end{bmatrix}$

$$C(A^T) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \cdots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\} = R^n \text{ , } N(A)=\{0\}$$

(c) $m \geq n$ ($\because$ If $m < n$, columns of A is linearly dependent)

(d) No

If $m > n = \text{rank}(A)$, it could be zero or one solution.

$\Rightarrow$ Ax=b is not always solvable.

(e)Yes

$N(A)=\{0\}$ imply it is uniqueness.

5.

Suppose $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$

$$A\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & 1 \\ a_{21} & a_{22} & a_{23} & 2 \\ a_{31} & a_{32} & a_{33} & 3 \end{bmatrix}$$

$$A\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow A=\begin{bmatrix} -1 & a_{12} & a_{13} & 1 \\ -2 & a_{22} & a_{23} & 2 \\ -3 & a_{32} & a_{33} & 3 \end{bmatrix}$$

$$A\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow A=\begin{bmatrix} -1 & 2 & a_{13} & 1 \\ -2 & 4 & a_{23} & 2 \\ -3 & 6 & a_{33} & 3 \end{bmatrix}$$

$$A\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow A=\begin{bmatrix} -1 & 2 & -1 & 1 \\ -2 & 4 & -2 & 2 \\ -3 & 6 & -3 & 3 \end{bmatrix}$$

6.

$$[R \quad E] = \left[\begin{array}{cccc|ccc} 1 & 2 & 0 & -1 & 2 & 4 & -5 \\ 0 & 0 & 1 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|ccc} 5 & 10 & -9 & -32 & 1 & 0 & 0 \\ -1 & -2 & 2 & 7 & 0 & 1 & 0 \\ 1 & 2 & -2 & -7 & 0 & 0 & 1 \end{array}\right] = [A \quad I_3]$$

(a)

$$C(A) = \text{span}\left\{\begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} -9 \\ 2 \\ -2 \end{bmatrix}\right\},\ C(A^T) = C(R^T) = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}\right\},$$

$$N = \begin{bmatrix} -2 & 1 \\ 1 & 0 \\ 0 & -3 \\ 0 & 1 \end{bmatrix},\ N(A) = \text{span}\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \end{bmatrix}\right\}$$

$$EA = R = \begin{bmatrix} 2 & 4 & -5 \\ 1 & 2 & -3 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 5 & 10 & -9 & -32 \\ -1 & -2 & 2 & 7 \\ 1 & 2 & -2 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ N(A^T) = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$$

(b)

$$\because 2\begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -9 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in C(A) \quad \therefore \text{It is solvable.}$$

(c)

$$\because \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \notin C(A) \quad \therefore \text{It is not solvable.}$$

(d)

$$\begin{bmatrix} A & A \\ A & A \end{bmatrix} \rightarrow \begin{bmatrix} A & A \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} R & R \\ 0 & 0 \end{bmatrix} \Rightarrow \text{Rank} = 2$$

7.

(a)False

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix},\ B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},\ R_A = R_B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \text{But} C(A) \neq C(B)$$

(b)True

$$C(A^T) = C({R_A}^T) = C({R_B}^T) = C(B^T)$$

(c)False

$A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$, $A^2=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$, $N_A=\begin{bmatrix}x\\ 0\end{bmatrix}\neq\begin{bmatrix}a\\ b\end{bmatrix}=N_{A^2}$

(d)True

A is invertible $\Rightarrow$ A is full rank $\Rightarrow$ N(A)={0}

A is invertible $\Rightarrow$ A^2 is invertible $\Rightarrow$ A^2 is full rank $\Rightarrow$ $N(A^2)=\{0\}$

$\therefore N(A)=N(A^2)=\{0\}$

(e)True

Suppose $A\neq 0$

$\Rightarrow$ Reduced row echelon form of A won't be 0

$\Rightarrow$ Rank(A) $\neq 0$

$\therefore$ If Rank(A)=0,then A=0

8.

(a)

B=EA，E is invertible $\Rightarrow$ $A=E^{-1}B$

$\Rightarrow a_j=E^{-1}b_j=E^{-1}(c_1b_1+c_2b_2+\cdots+c_nb_n)=c_1E^{-1}b_1+c_2E^{-1}b_2+\cdots+c_nE^{-1}b_n=c_1a_1+c_2a_2+\cdots+c_na_n$

(b) No

Counter example:$E=\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$, $A=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$, $B=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$

$b_1=c_1b_1+c_2b_2=\begin{bmatrix}1\\ 1\end{bmatrix}$ 令$c_1=0$ $c_2=1$

$\Rightarrow a_1=\begin{bmatrix}1\\ 0\end{bmatrix}\neq c_1a_1+c_2a_2=a_2=\begin{bmatrix}1\\ 1\end{bmatrix}$

(c)

$$A=\begin{bmatrix}5 & 10 & -9 & -32\\ -1 & -2 & 2 & 7\\ 1 & 2 & -2 & -7\end{bmatrix}$$

$a_2=2a_1$

$a_4=-a_1+3a_3$