**Linear Algebra**

**Problem Set 4 Solution** **Spring 2015**

**1.(10pts)**

假設以下矩陣皆為 $m \times n$

(a) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ ,C(A) = span{$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$,$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$} , C($A^T$)=span{$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$,$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$}

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in$ C($A^T$), because $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

(c) **impossible**, the matrix is 3 by 2, by rank-nullity theorem, n = rank + nullity.
rank $\geq 1$, dimN $\geq 2$, rank + nullity $\geq 3 \neq 2$

(d) **impossible**, the matrix is m by 3, n = 3.
Row space has basis [1 0 0] $\Rightarrow$ rank =1.
nullspace has basis $[1\ 1\ 1]^T$ $\Rightarrow$ dimN =1.
by rank-nullity theorem, rank+ dimN=1 + 1 = 2 $\neq n$

(e) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, C(A) = span{$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$}=R(A)

R = $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, N(A) = span{$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$} = N($A^T$)

**2.(15pts)**

(a)

$$\begin{bmatrix} 1 & 2 & 5 & 0 \\ 1 & 3 & 7 & 1 \\ 1 & 3 & 7 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 5 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b)

Let $x_3 = \alpha$, $x_4 = \beta$.

$x_1 = -\alpha + 2\beta$, $x_2 = -2\alpha - \beta$

$$x=\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}\alpha + \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}\beta$$

(c)

$$N(A) = \text{span}\{\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}\}, \text{ nullspace matrix of A} = \begin{bmatrix} -1 & 2 \\ -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(d)

$$\text{augmented matrix } \begin{bmatrix} 1 & 2 & 5 & 0 & b_1 \\ 1 & 3 & 7 & 1 & b_2 \\ 1 & 3 & 7 & 1 & b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 5 & 0 & b_1 \\ 1 & 3 & 7 & 1 & b_2 \\ 0 & 0 & 0 & 0 & b_3 - b_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 5 & 0 & b_1 \\ 0 & 1 & 2 & 1 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 \end{bmatrix}$$

if Ax=b is consistent, then $b_3 - b_2 = 0$, $b_3 = b_2$.

(e)

$Ax_p = b$ $\Rightarrow$ set free varible to be 0.

$$x_p = \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{bmatrix}, \quad A\begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 11 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 5 & 0 \\ 1 & 3 & 7 & 1 \\ 1 & 3 & 7 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 11 \end{bmatrix}$$

$\Rightarrow$

$x_1 + 2x_2 = 8$

$x_1 + 3x_2 = 11$ $\Rightarrow$ $x_1 = 2$, $x_2 = 3$.

$$x = x_h + x_p = \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}\alpha + \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}\beta + \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$

**3.(10pts)**

First find row echelon form of each matrix

(1) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, rank = 2, m = 2, n = 3.

(2) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, rank = 1, m = 2, n = 3.

(3) $\begin{bmatrix} 1 & 2 \\ 0 & -2 \\ 0 & 0 \end{bmatrix}$, rank = 2, m = 3, n = 2.

(4) $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, rank = 2, m = 3, n = 3.

(5) $\begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$, rank = 2, m = 2, n = 2.

(a) (1)(5), At least one solution for every **b** $\Leftrightarrow$ full row rank $\Leftrightarrow$ r = m

(b) (3)(5), At most one solution for every **b** $\Leftrightarrow$ full column rank $\Leftrightarrow$ r = n

(c) (5), Exactly one solution for every **b** $\Leftrightarrow$ full rank $\Leftrightarrow$ r = n = m

(d) (1), Infinite many solutions for every **b** $\Leftrightarrow$ r < n, r = m

(e) (2)(3)(4), No solution for some **b** $\Leftrightarrow$ r < m

**4.(15pts)**

(a) rank(A) = m.

(b)

Because the rows of A are linearly independent, reduced row echelon form of A could be write as $\begin{bmatrix} I_m & | 0 \end{bmatrix}$, C(A) = span{$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$} = $R^m$, $N(A^T) = \{0\}$.

(c)

$n \geq m$(If n < m , rows of A are linearly dependent)

(d)

Yes, rank(A) = m.

(e)

No, N(A) is not always = {0}, the solution is not necessarily unique.

(f)

Yes

(g)

No

**5.(10pts)**

$$\mathbf{A}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}=\begin{bmatrix}1\\2\\1\end{bmatrix} \Rightarrow \mathbf{A}=\begin{bmatrix}1 & a_{12} & a_{13} & a_{14}\\ 2 & a_{22} & a_{23} & a_{24}\\ 1 & a_{32} & a_{33} & a_{34}\end{bmatrix}$$

$$\mathbf{A}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \mathbf{A}=\begin{bmatrix}1 & a_{12} & a_{13} & -1\\ 2 & a_{22} & a_{23} & -2\\ 1 & a_{32} & a_{33} & -1\end{bmatrix}$$

$$\mathbf{A}\begin{bmatrix}1\\1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \mathbf{A}=\begin{bmatrix}1 & -1 & a_{13} & -1\\ 2 & -2 & a_{23} & -2\\ 1 & -1 & a_{33} & -1\end{bmatrix}$$

$$\mathbf{A}\begin{bmatrix}1\\1\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \mathbf{A}=\begin{bmatrix}1 & -1 & 0 & -1\\ 2 & -2 & 0 & -2\\ 1 & -1 & 0 & -1\end{bmatrix}$$

**6.(15pts)**

(a)

Let B = $\begin{bmatrix} -1 & 5 & 2 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{bmatrix}$, $B^{-1}$ = $\begin{bmatrix} -1 & -11 & 8 \\ 0 & -3 & 2 \\ 0 & 2 & -1 \end{bmatrix}$

$B^{-1}\begin{bmatrix} A & I_3 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -11 & -45 & 1 & 0 & 0 \\ 0 & 0 & -3 & -12 & 0 & 1 & 0 \\ 0 & 0 & 2 & 8 & 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow$ **A =** $\begin{bmatrix} -1 & -2 & -11 & -45 \\ 0 & 0 & -3 & -12 \\ 0 & 0 & 2 & 8 \end{bmatrix}$

(b)

C(A) = span{$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 11 \\ 3 \\ -2 \end{bmatrix}$}, R(A) = span{$\begin{bmatrix} -1 \\ -2 \\ -11 \\ -45 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ -3 \\ -12 \end{bmatrix}$}

$x_1 = -2x_2 - x_4$

$x_3 = -4x_4$

N(A)=span{$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \\ -4 \\ 1 \end{bmatrix}$}

$A^T$=$\begin{bmatrix} -1 & 0 & 0 \\ -2 & 0 & 0 \\ -11 & -3 & 2 \\ -45 & -12 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -3 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 = 0$

$-3x_2 = -2x_3$

N($A^T$)=span{$\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$}

The basis for column space of A= $\{\begin{bmatrix}-1\\0\\0\end{bmatrix},\begin{bmatrix}-11\\-3\\2\end{bmatrix}\}$ ,

the basis for row space of A = $\{\begin{bmatrix}-1\\-2\\-11\\-45\end{bmatrix},\begin{bmatrix}0\\0\\1\\4\end{bmatrix}\}$,

the basis for nullspace of A = $\{\begin{bmatrix}-2\\1\\0\\0\end{bmatrix},\begin{bmatrix}-1\\0\\-4\\1\end{bmatrix}\}$,

the basis for left nullspace of A = $\{\begin{bmatrix}0\\2\\3\end{bmatrix}\}$

(c)

No, $\begin{bmatrix}0\\1\\1\end{bmatrix}\notin C(A)$

(d)

Yes, $\begin{bmatrix}1\\0\\0\end{bmatrix}\in C(A)$

**7.(15pts)**

(a) B = EA , E is invertible $\Rightarrow$ $A = E^{-1}B$

$\Rightarrow a_j = E^{-1}b_j = E^{-1}(c_1b_1 + c_2b_2 + \cdots + c_nb_n) = c_1E^{-1}b_1 + c_2E^{-1}b_2 + \cdots + c_nE^{-1}b_n = c_1a_1+c_2a_2+\cdots+c_na_n$

(b) NO

Counter example: $E=\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$, $A = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$, $B = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$

$b_1 = c_1b_1 + c_2b_2 = \begin{bmatrix}1\\1\end{bmatrix}$ Let $c_1 = 0, c_2 = 1$

$\Rightarrow a_1 = \begin{bmatrix}1\\0\end{bmatrix} \neq c_1a_1+c_2a_2 = a_2 = \begin{bmatrix}1\\1\end{bmatrix}$

(c) $\mathbf{A} = \begin{bmatrix}-1 & -2 & -11 & -45\\0 & 0 & -3 & -12\\0 & 0 & 2 & 8\end{bmatrix}$

$a_2 = 2a_1$

$a_4 = 4a_3 + a_1$

**8.(10pts)**

(a) False

$A=\begin{bmatrix}1 & 1 & 1\\2 & 2 & 2\\3 & 3 & 3\end{bmatrix}$, $B=\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}$, $R_A = R_B = \begin{bmatrix}1 & 1 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$, But $C(A) \neq C(B)$.

(b) True

$C(A^T) = C(R_A{}^T) = C(R_B{}^T) = C(B^T)$

(c) False

$A = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$, $A^2 = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$, $N(A) = \text{span}\{\begin{bmatrix}1\\0\end{bmatrix}\}$, $N(A^2) = \text{span}\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\}$

(d) True

A is invertible $\Rightarrow$ A is full rank $\Rightarrow$ N(A)={0}

If A is invertible ,det(A) $\neq 0$

$\det(A^2) = \det(A)\det(A) \neq 0$

$\Rightarrow A^2$ is invertible $\Rightarrow A^2$ is full rank $\Rightarrow N(A^2)=\{0\}$

$N(A) = N(A^2)$

(e) Flase

Let $A = \begin{bmatrix} 5 & -3 & 2 \\ 15 & -9 & 6 \\ 10 & -6 & 4 \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\text{rank}(A^2) = 0$, $A \neq 0$.