Linear Algebra

Problem Set 4 Solution **Spring 2016**

1. (15pts)

(1) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$ (Full row rank, $r = 2$)

(2) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $(r = 1)$

(3) $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ (Full column rank, $r = 2$)

(4) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $(r = 2)$

(5) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (Full rank, $r = 3$)

(a) (1)(5)

(b) (3)(5)

(c) (5)

(d) (1)

(e) (2)(3)(4)

2.(20pts)

$$[A \quad | I_5] = \left[\begin{array}{cccc|ccccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 & 0 \\ 5 & 6 & 7 & 8 & 0 & 1 & 0 & 0 & 0 \\ 9 & 10 & 11 & 12 & 0 & 0 & 1 & 0 & 0 \\ 13 & 14 & 15 & 16 & 0 & 0 & 0 & 1 & 0 \\ 17 & 18 & 19 & 20 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|ccccc} 1 & 0 & -1 & -2 & -\frac{3}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & \frac{5}{4} & -\frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & -4 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|c} I_r & F & E_1 \\ 0 & 0 & E_2 \end{array}\right]$$

Nullspace matrix $N = \begin{bmatrix} 1 & 2 \\ -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

(a) $\begin{bmatrix} 1 \\ 5 \\ 9 \\ 13 \\ 17 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 10 \\ 14 \\ 18 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & -1 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(e)

$$Ax = b \rightarrow EAx = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} -\frac{3}{2} b_1 + \frac{1}{2} b_2 \\ \frac{5}{4} b_1 - \frac{1}{4} b_2 \\ b_1 - 2b_2 + b_3 \\ 2b_1 - 3b_2 + b_4 \\ 3b_1 - 4b_2 + b_5 \end{bmatrix} = Eb$$

$$\begin{cases} b_1 - 2b_2 + b_3 = 0 \\ 2b_1 - 3b_2 + b_4 = 0 \\ 3b_1 - 4b_2 + b_5 = 0 \end{cases}$$

(f)

$$[A \quad |b] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 5 & 6 & 7 & 8 & 4 \\ 9 & 10 & 11 & 12 & 8 \\ 13 & 14 & 15 & 16 & 12 \\ 17 & 18 & 19 & 20 & 16 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 2 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = [R \quad |c]$$

$$x_h = c_1 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \qquad x_p = d_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix} + d_2 \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$$Rx_p = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left(d_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix} + d_2 \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = c$$

$$\begin{cases} 6d_1 - 8d_2 = 2 \\ -8d_1 + 14d_2 = -1 \end{cases} \rightarrow d_1 = 1, d_2 = 1/2 \rightarrow x_p = \frac{1}{2} \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\boldsymbol{x_g = x_h + x_p = c_1 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}}$$

3.(20pts)

(a) A has n independent columns , $\mathbf{rank}(\boldsymbol{A}) = \mathbf{n}$

(b) $\mathbf{n} \leq \mathbf{m}$. (if $\mathrm{m} < \mathrm{n}$, rows of A are linearly independent)

(c) $\left[\dfrac{I_n}{0_{(m-n)\times n}}\right]$

(d) $\mathrm{R}(A) = \mathrm{span}\{[1,0,\cdots,0],[0,1,\cdots,0],\cdots,[0,0,\cdots,1]\} = R^n$

$\mathrm{N}(A) = \{0\}$

(e) $\dim \mathrm{C}(A) = \mathrm{rank}(A) = \mathrm{n}$

(f) $\dim \mathrm{N}(A^T) = \mathrm{m} - \mathrm{rank}(A^T) = \mathrm{m} - \mathrm{n}$

(g) False, A is full column rank, it has zero or one solution.

(h) Let $Ax_1 = b, Ax_2 = b, and \ x_1 \neq x_2$.

$Ax_1 - Ax_2 = A(x_1 - x_2) = 0$

A is full column rank, so $\mathrm{N}(A) = \{0\}$.

That is, $x_1 - x_2 = 0$. It is contradiction.

Hence, the answer is unique.

(i) $B = A^T(AA^T)^{-1}$

$\mathrm{rank}(AA^T) = \mathrm{rank}(A) = \mathrm{n}$

When $\mathrm{m} \neq \mathrm{n}$, it doesn't exist $(AA^T)^{-1}$, hence, B doesn't exist.

(j) $B = (A^TA)^{-1}A^T$

$\mathrm{rank}(A^TA) = \mathrm{rank}(A) = \mathrm{n}$

There exists $(A^TA)^{-1}$ let A has a left inverse B.

4.(15pts)

(a)

Suppose $x \in \mathrm{N}(A)$

$Ax = 0, \quad A(Ax) = A \cdot 0 \rightarrow A^2x = 0 \rightarrow x \in \mathrm{N}(A^2)$

$\therefore \mathrm{N}(A) \subseteq \mathrm{N}(A^2) \rightarrow$ not necessarily equal.

$\rightarrow \mathrm{N}(A) \subseteq \mathrm{N}(A^2) \subseteq \mathrm{N}(A^3) \ldots$.

(b)

Suppose $y \in \mathrm{C}(A^k)$

$A^kx = y \rightarrow A^{k-1}(Ax) = y \rightarrow y \in \mathrm{C}(A^{k-1})$

$\therefore \mathrm{C}(A^{k-1}) \supseteq \mathrm{C}(A^k)$

$\rightarrow \mathrm{C}(A) \supseteq \mathrm{C}(A^2) \supseteq \mathrm{C}(A^3) \ldots \supseteq \mathrm{C}(A^k)$

(c)

$\mathrm{N}(A^2) = \mathrm{N}(A^3)$

Suppose $x \in \mathrm{N}(A^4)$

$A^4x = 0 \rightarrow A^3(Ax) = 0 \rightarrow Ax \in \mathrm{N}(A^3) = \mathrm{N}(A^2)$

That is, $A^2(Ax) = A^3x = 0 \rightarrow x \in \mathrm{N}(A^3)$

$\therefore \mathrm{N}(A^3) = \mathrm{N}(A^4)$

5.(5pts)

$C = \begin{bmatrix} A \\ B \end{bmatrix}$

Suppose $x \in \mathrm{N}(C)$

$Cx = 0 \rightarrow \begin{bmatrix} A \\ B \end{bmatrix} x = \begin{bmatrix} Ax \\ Bx \end{bmatrix} = 0$

$\mathrm{N}(C) = \mathrm{N}(A) \cap \mathrm{N}(B)$

6.(5pts)

Columns of A and B are linearly independent, so N(A)=0, N(B)=0.
Suppose $(AB)x = 0 \rightarrow A(Bx) = 0$, hence x must be zero.
That is, columns of AB are linearly independent.

7.(5pts)

u, v, w are linearly independent vectors.
$c_1 u + c_2 v + c_3 w = 0, (c_1 = c_2 = c_3 = 0)$

$d_1 u + d_2(u + v) + d_3(u + v + w) = 0$
$\rightarrow (d_1 + d_2 + d_3)u + (d_2 + d_3)v + d_3 w = 0$
$\rightarrow d_1 = d_2 = d_3 = 0$
$u, u + v, u + v + w$ are linearly independent vectors.

8.(15pts)

B and C are invertible matrix, so columns of them are linearly independent .

(a)
Suppose $x \epsilon \mathrm{N}(AB), ABx = 0$
Because columns of A are linearly independent, so it exists left inverse A'.
$A'ABx = 0 \rightarrow Bx = 0 \rightarrow B^{-1}Bx = 0 \rightarrow x = 0$
Hence, columns of AB are linearly independent.

(b)
Suppose $x \epsilon \mathrm{N}(CA), CAx = 0$
$C^{-1}CAx = 0 \rightarrow Ax = 0 \rightarrow x = 0$
Hence, columns of CA are linearly independent.