Solution to Problem Set 5

1.

(a)

Suppose $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ and $\mathbf{x} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_n\mathbf{v}_n$

By subtraction $(c_1 - d_1)\mathbf{v}_1 + (c_2 - d_2)\mathbf{v}_2 + \cdots.(c_n - d_n)\mathbf{v}_n = \mathbf{0}$

From the independence of the v' s, each $c_i - d_i = 0$

Hence $c_i = d_i$, and there are not two ways to produce $\mathbf{v}$

(b)

Suppose $c_1\mathbf{A}\mathbf{v}_1 + c_2\mathbf{A}\mathbf{v}_2 + ... + c_n\mathbf{A}\mathbf{v}_n = 0$

$\rightarrow A\sum_{i=1}^{n} c_i v_i = 0$

∵ For any A , A has inverse

$\therefore \sum_{i=1}^{n} c_i v_i = 0$

$\rightarrow c_1 = c_2 = ... = 0$

$\rightarrow \mathbf{A}\mathbf{v}_1, \mathbf{A}\mathbf{v}_2 ... \mathbf{A}\mathbf{v}_n$ is also a basis for $\mathbf{R}^n$

2.

(a)

B = EA

$\rightarrow$ $A = E^{-1}B$

$\rightarrow$ $\mathbf{a}_j = E^{-1}\mathbf{b}_j$

If $\mathbf{b}_j = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + ... + c_n\mathbf{b}_n$

$\mathbf{a}_j = E^{-1}(c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + ... + c_n\mathbf{b}_n)$

$= c_1(E^{-1}\mathbf{b}_1) + c_2(E^{-1}\mathbf{b}_2) + ... + c_n(E^{-1}\mathbf{b}_n)$

$= c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + ... + c_n\mathbf{a}_n$

(b)

Let $R = \begin{bmatrix} 1 & 5 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix}$

∵ $\mathbf{a}_2 = 5 \times \mathbf{a}_1$

$\mathbf{a}_4 = 2 \times \mathbf{a}_1 + (-3) \times \mathbf{a}_3$

∴ $\mathbf{a}_2$ and $\mathbf{a}_4$ are linear combination of $\mathbf{a}_1$ and $\mathbf{a}_3$

(c)

Counterexample

$$\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} \quad b_1 = b_2 \text{ , but } a_1 \neq a_2$$

3.

The reduced row echelon form of A is $\begin{bmatrix}1 & -2 & 3 & 0 & -2\\ 0 & 0 & 0 & 1 & 3\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$

→ From the reduced row echelon form , we can get the row space and the null space of A.

∴ span{$[1\ -2\ 3\ 0\ -2]^T$, $[0\ 0\ 0\ 1\ 3]^T$} is the row space

→ the nullspace matrix is $\begin{bmatrix}2 & -3 & 2\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -3\\ 0 & 0 & 1\end{bmatrix}$

∴ span {$[2\ 1\ 0\ 0\ 0]^T$, $[-3\ 0\ 1\ 0\ 0]^T$, $[2\ 0\ 0\ -3\ 1]^T$} is the null space

To find the column space and the left column space , we recover A first.

A = E × (row echelon form of A) and I_3 = E × $\begin{bmatrix}1 & 1 & 1\\ 2 & 3 & 3\\ 1 & 1 & 0\end{bmatrix}$

$$\rightarrow E = \begin{bmatrix}1 & 1 & 1\\ 2 & 3 & 3\\ 1 & 1 & 0\end{bmatrix}^{-1} = \begin{bmatrix}3 & -1 & 0\\ -3 & 1 & 1\\ 1 & 0 & -1\end{bmatrix}$$

∴ A = E × (row echelon form of A) = $\begin{bmatrix}3 & -6 & 9 & -1 & -9\\ -3 & 6 & -9 & 1 & 9\\ 1 & -2 & 3 & 0 & -2\end{bmatrix}$

∵ The row operation doesn' t change the relation between the columns of A

→$column_2$ = $column_1$ × (-2)

$column_3$ = $column_1$ × 3

$column_5$ = $column_1$ × (-2) + $column_4$ × 3

∴span {$[3\ -3\ 1]^T$,$[-1\ 1\ 0]^T$} is the column space of A

∵E^{-1}× A =(row echelon form of A)

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 3 & -6 & 9 & -1 & -9 \\ -3 & 6 & -9 & 1 & 9 \\ 1 & -2 & 3 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 & 0 & -2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We focus on the zero row of echelon form of A

$$\rightarrow [\ 1\ 1\ 0] \times \begin{bmatrix} 3 & -6 & 9 & -1 & -9 \\ -3 & 6 & -9 & 1 & 9 \\ 1 & -2 & 3 & 0 & -2 \end{bmatrix} = 0$$

$\rightarrow [1\ 1\ 0]^T \in$ the left null space of A

We know that the dimention of left null space = the number of rows – rank , and rank(A)= 2

$\rightarrow$ the dimention of left null space = 1

$\therefore$ span{$[1\ 1\ 0]^T$} is the left null space

4

(a)

A subspace in $\mathbf{R}^4$ = {$\mathbf{x}=[x_1\ x_1\ x_1\ x_1]^T \mid \mathbf{x} \in \mathbf{R}^4$}

$\rightarrow$Any vectors in this subspace are $[x_1\ x_1\ x_1\ x_1]^T = x_1[1\ 1\ 1\ 1]^T$

$\rightarrow x_1$ is any constant , so the vectors in this subspace are the linear combinations of $[\ 1\ 1\ 1\ 1]^T$, $\therefore$ The basis is{$[1\ 1\ 1\ 1]^T$

(b)

A subspace in $\mathbf{R}^4$ = {$\mathbf{x} = [x_1\ x_2\ x_3\ x_4]^T \mid x_1 + x_2 + x_3 + x_4 = 0, \mathbf{x} \in \mathbf{R}^n$ }

$\rightarrow$Let $x_2 = C_1$, $x_3 = C_2$, $x_3 = C_3$, $x_4 = C_3$ and $C_1, C_2, C_3 \in \mathbf{R}$

$\rightarrow x_1 = -C_1 - C_2 - C_3$

$$\rightarrow x = C_1 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\rightarrow$x is the linear combination of {$[-1\ 1\ 0\ 0\ ]^T$, $[-1\ 0\ 1\ 0\ ]^T$,$[-1\ 0\ 0\ 1]^T$}

$\therefore$ The basis is {$[-1\ 1\ 0\ 0\ ]^T$, $[-1\ 0\ 1\ 0\ ]^T$,$[-1\ 0\ 0\ 1]^T$}

(c)

Let $u_1 = [1\ 1\ 1\ 0]^T$ and $u_2 = [1\ 0\ 1\ 1]^T$

A subspace in R^4 = { $x=[x_1\ x_2\ x_3\ x_4]^T \mid x^T u_1 = x^T u_2 = 0$, $x \in R^4$ }

$\rightarrow$We can rewrite the relation between $\mathbf{x}$, $\mathbf{u}_1$ and $\mathbf{u}_2$ as

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = 0$$

→This subspace is equal to the left null space of $\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$

→The left nullspace matrix = $\begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

∴The basis is {[-1 1 0 1 $]^T$,[-1 0 1 0 $]^T$}

(d)

A subspace in R^4 = { x = [a b c d $]^T$ | a = b and c = d , x ∈ R^4}

→x = [a b c d $]^T$ = [a a c c$]^T$ = a × [1 1 0 0] + c × [0 0 1 1]

∴ The basis is{ [1 1 0 0 $]^T$, [0 0 1 1 $]^T$}

5.

(a)

From nullspace basis $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in $\mathbf{R}^3$

→The number of columns is 3 = rank + Dimention of null space

But dim(N) = 1 , dim(C) = 1

→rank + Dimention of null space = 2 ≠ 3

→This matrix is impossible

(b)

Ans: $\begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}$

The column space is span{[1 2 0 $]^T$, [0 0 1$]^T$} which contains {[1 2 0 $]^T$, [0 0 2 $]^T$}

The row space is span{[1 0 $]^T$, [0 1$]^T$} which contains {[1 2 $]^T$, [3 4 $]^T$ }

(c)

Ans: $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

Dimension of column space = dimension of nullspace = 1

(d)

$$\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ a & b \end{bmatrix} = 0$$

→ a = -1 , b = -1

Ans : $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

(e)

Row space = column space → The matrix must be square

→Dimention of null space = the number of columns － rank

Dimention of left null space = the number of rows － rank

But the matrix is square and dimention of column space = dimention of row space.

∴This matrix is impossible

6.

(a)

dim N(A) = 1 (There is one homogeneous solution)

It is obvious that A is 4 by 3.

From the rank theorem , rank(A) = n- dim N(A) = 2

(b)

Since dim C(A) = rank(A) = 2 , a basis for the column space of A contains two vector.

∵ $Ax = [\ 2\ 2\ 2\ 2\ ]^T$ is sovable ∴$[\ 2\ 2\ 2\ 2\ ]^T$ is in C(A)

∵ $Ax = [\ 2\ 2\ 1\ 1\ ]^T$ is sovable ∴$[\ 2\ 2\ 1\ 1\ ]^T$ is in C(A)

→basis of C(A) = {$[\ 2\ 2\ 2\ 2\ ]^T$, $[\ 2\ 2\ 1\ 1\ ]^T$ }

(c)

We know that N(A) = $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Recall the method of finding N(A) from the reduced echelon form of A.

(If A ~ $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$, then N(A) = $\begin{bmatrix} -F \\ I \end{bmatrix}$)

$\therefore$ We can get A ~ $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$