2013 spring HW5

1.

(a) $T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Ax$, $\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, N_A = 0$

$R(T) = C(A) = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$, Rank(T) = the dimension of R(T) = 2

$\ker(T) = N(A) = \{0\}$

(b)$T(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = Ax$, $\begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, N_A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

$R(T) = C(A) = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$, Rank(T) = the dimension of R(T) = 1

$\ker(T) = N(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

(c) $T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Ax$

$R(T) = C(A) = \{0\}$, Rank(T)=the dimension of R(T) = 0

$\ker(T) = N(A) = R^2 = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

2.

(a) T is one to one

$\Rightarrow$ Each vector in R^n will be mapped to only one vector in R^m

$\Rightarrow$ The null space of must be the zero vector , because only one vector can be mapped to zero

$\Rightarrow$ A is full column rank

$\Rightarrow r = n \le m$

(b) T maps V onto W

$\Rightarrow$ All vectors in R^m will be mapped

$\Rightarrow R(T) = C(A) = R^m$

$\Rightarrow$ A is full row rank

$\Rightarrow r = m \le n$

(c) T is one to one and maps V onto W

$\Rightarrow$ A is full rank

$\Rightarrow$ r = m = n

3.

(a)

$\because \{v_1, v_2, v_3\}$ is linearly independent. $\therefore c_1v_1 + c_2v_2 + c_3v_3 = 0$, $c_1 = c_2 = c_3 = 0$

$\because$ T is one-to-one $\therefore T(c_1v_1 + c_2v_2 + c_3v_3) = T(0) = 0$

$\because$ T is a linear transform.

$\therefore T(c_1v_1 + c_2v_2 + c_3v_3) = T(c_1v_1) + T(c_2v_2) + T(c_3v_3) = c_1T(v_1) + c_2T(v_2) + c_3T(v_3) = 0$

$\therefore c_1 = c_2 = c_3 = 0$

$\Rightarrow \{T(v_1), T(v_2), T(v_3)\}$ is linearly independent.

(b)

$\because \{v_1, v_2, v_3\}$ is linearly dependent. $\therefore$ Let $v_1 = c_2v_2 + c_3v_3, c_2 \neq 0$ or $c_3 \neq 0$

$\because$ T is a linear transform.

$\therefore T(v_1) = T(c_2v_2 + c_3v_3) = c_2T(v_2) + c_3T(v_3), c_2 \neq 0$ or $c_3 \neq 0$

$\Rightarrow \{T(v_1), T(v_2), T(v_3)\}$ is linearly dependent.

(c)

$\because \{T(u), T(v)\}$ is linearly dependent. $\therefore$ Let $aT(u) + bT(v)=0, a \neq 0$ or $b \neq 0$

$\because a \neq 0$ or $b \neq 0$ $\therefore au + bv \neq 0$

$\Rightarrow$ Choose $x=au + bv$, $T(x) = T(au + bv) = aT(u) + bT(v)=0$

$\Rightarrow$ There exists a nonzero vetor x such that T(x)=0

(d)No

If {u,w} is linearly dependent , {u,v,w} is not linearly independent.

4.

(a)

$2 - t + t^2 = c_1(1 + t) + c_2(1 + t^2) + c_3(t + t^2)$

$\Rightarrow c_1 = 0, c_2 = 2, c_3 = -1$

$$\Rightarrow [P]_B = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$

(b)

Suppose basis = $\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\}$

The coordinate vector of $\begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}$ is [1 -1 0 0]

The coordinate vector of $\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$ is [1 0 0 0]

The coordinate vector of $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ is [0 1 1 0]

The coordinate vector of $\begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}$ is [0 1 0 1]

Let matrix be $\begin{bmatrix}1 & 0 & 0 & 0\\1 & -1 & 0 & 0\\0 & 1 & 1 & 0\\0 & 1 & 0 & 1\end{bmatrix} \rightarrow \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$

The row are independent.

$\therefore \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}$ are linear independent.

5.

(a) $\begin{bmatrix}5\\-1\end{bmatrix} = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix} \Rightarrow c_1 = 2,\ c_2 = 3$

$$\mathrm{T}(5,-1) = c_1\mathrm{T}(1,1) + c_2\mathrm{T}(1,2) = 2\begin{bmatrix}1\\2\end{bmatrix} + 3\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}8\\13\end{bmatrix}$$

(b) $\mathrm{T}(x,y) = c_1\mathrm{T}(1,1) + c_2\mathrm{T}(1,2) = c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}3\\5\end{bmatrix} \Rightarrow c_1 = 1,\ c_2 = 1$

$$c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix} \Rightarrow x = 2,\ y = 0$$

(c)

$$[\mathrm{T}]_s = \begin{bmatrix}1 & 2\\2 & 3\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}^{-1} = \begin{bmatrix}\frac{3}{2} & \frac{-1}{2}\\ \frac{5}{2} & \frac{-1}{2}\end{bmatrix}$$

(d)

$$[T]_B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{3}{2} & \frac{-1}{2} \\ \frac{5}{2} & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{5}{2} \\ \frac{-1}{2} & \frac{-1}{2} \end{bmatrix}$$

(e)

$$[T]_C = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{3}{2} & \frac{-1}{2} \\ \frac{5}{2} & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & \frac{17}{5} \\ \frac{-1}{10} & \frac{1}{5} \end{bmatrix}$$

6.

(a)

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} = \{v_1, v_2, v_3, v_4\} \quad , \quad [T]_S = \{T(v_1), T(v_2), T(v_3), T(v_4)\}$$

$$T(v_1) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad , \quad T(v_2) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 1 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T(v_3) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = 2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 4 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad , \quad T(v_4) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[T]_S = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix}$$

(b)

$$B = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right\} = \{v_1, v_2, v_3, v_4\} \quad , \quad [T]_B = \{T(v_1), T(v_2), T(v_3), T(v_4)\}$$

$$T(v_1) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 5 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - 2 \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \quad , \quad T(v_2) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = 5 \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} - 2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$T(v_3) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad , \quad T(v_4) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = - \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$[T]_B = \begin{bmatrix} 5 & 0 & -1 & 0 \\ 0 & 5 & 0 & -1 \\ -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix}$$