Linear Algebra

Problem Set 5 Solution **Spring 2015**

1.(15pts)

(a)

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$$, the standard matrix of T is $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$, rank(T) = 2.

N(T) = {0}, the bases for nullspace of T is {0}.

R(T) = span$\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$, the bases for range of T is $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$.

(b)

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_3 \end{bmatrix}$$, the standard matrix of T is $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, rank(T) = 2.

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, let x_3=c, $x_1 = -x_3$, $x_2 = -x_3$, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$=c$\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

N(T) =span$\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right\}$, the bases for nullspace of T is $\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right\}$.

R(T) = span$\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right\}$, the bases for range of T is $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right\}$.

(c)

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \\ x_2 \end{bmatrix}$$, the standard matrix of T is $\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, rank(T) = 2.

N(T) = {0}, the bases for nullspace of T is {0 }.

R(T) = span$\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\}$, the bases for range of T is $\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\}$.

2.(20pts)

(a)

T is one to one $\rightarrow$ N(T) = {0}

$\Rightarrow$ dim(V) = dim(N(T)) + dim(R(T)), dim(R(T)) = rank(T)

$\Rightarrow$ n = 0 + r $\rightarrow$ n = $r \leq m$

(b)

T maps V onto W,

$\Rightarrow$ R(T) = W $\Rightarrow$ dim(R(T)) = dim(W).

$\Rightarrow$ dim(R(T)) = rank(T), r = $m \leq n$

(c)

From (a) and (b), if T is one to one and T maps V onto W

$\Rightarrow$ n = $r \leq m$ and r = $m \leq n$

$\Rightarrow$ n = r = m

(d)

rank(T) = dim(R(T)), r = n - nullity(T)

$\Rightarrow m > r$, $n > r$

3.(15pts)

(a) True.

Let $x = c_1v_1 + c_2v_2 + c_3v_3 \neq 0$

$T(x) = T(c_1v_1 + c_2v_2 + c_3v_3) = c_1T(v_1) + c_2T(v_2) + c_3T(v_3) = 0$

because $T(x) = 0$, $T(v_1) = T(v_2) = T(v_3) = 0$.

$\Rightarrow$ c_1, c_2, c_3 is not always 0

$\Rightarrow$ $\{T(v_1), T(v_2), T(v_3)\}$ is linearly dependent.

(b) True.

$\{v_1, v_2, v_3\}$ is linearly dependent

$\Rightarrow$ Let $v_1 = c_2v_2 + c_3v_3$, where $c_2 \neq 0$ and $c_3 \neq 0$.

T is a linear transform

$\Rightarrow T(v_1) = T(c_2v_2 + c_3v_3) = c_2T(v_2) + c_3T(v_3)$,where $c_2 \neq 0$ and $c_3 \neq 0$.

$\Rightarrow T(v_1)$ is a linear combination of $T(v_2)$ and $T(v_3)$.

$\Rightarrow \{T(v_1), T(v_2), T(v_3)\}$ is linearly dependent.

(c) False

for counter example, let $u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $w = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\{u,v\}$, $\{v,w\}$, $\{u,w\}$ is linearly independent

but $\{u, v, w\}$ is not linearly independent($w = u + v$)

4.(10pts)

(a)

$p(t) = c_1p_1(t)+c_2p_2(t)+c_3p_3(t) = (c_1+c_2+c_3) + (c_2+2c_3)t + (c_3)t^2$

$= 2 + t + 3t^2$

$\Rightarrow$ $c_1+c_2+c_3 = 2$, $c_2+2c_3 = 1$, $c_3 = 3$.

$\Rightarrow c_1 = 4$, $c_2 = -5$, $c_3 = 3$

$[p(t)]_B = \begin{bmatrix} 4 \\ -5 \\ 3 \end{bmatrix}$

(b)

Suppose basis = $\{1, t, t^2\}$

$p_1(t) = 1 + t^2 \Rightarrow$ [1 0 1]

$p_2(t) = 1 + t + 2t^2 \Rightarrow$ [1 1 2]

$p_3(t) = 2 + t + 3t^3 \Rightarrow$ [2 1 3]

[2 1 3] = [1 0 1] + [1 1 2] $\Rightarrow$ $p_1(t)$, $p_2(t)$ and $p_3(t)$ are linearly dependent.

5.(10pts)

(a)

Let a = $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, v_1 = $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, v_2 = $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$a = c_1 v_1 + c_2 v_2 \Rightarrow c_1 - c_2 = 1, c_1 + c_2 = 3 \Rightarrow c_1 = 2, c_2 = 1$

$T(a) = T(2v_1)+T(v_2) = 2T(v_1) + T(v_2) = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

(b)

Let b = T(x,y) = $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$, u_1 = $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, u_2 = $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$b = c_3 u_1 + c_4 u_2$, $c_3 + 3c_4 = 3$, $2c_3 + 4c_4 = 5 \Rightarrow c_3 = 1.5$, $c_4 = 0.5$

$b = T(x,y) = 1.5u_1 + 0.5u_2 = 1.5T(1,1) + 0.5T(-1,1) = T(1.5,1.5)+T(-0.5,0.5)$

$= T(1, 2)$

$\Rightarrow x = 1, y = 2$

(c)

$$T\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$T\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$T = \begin{bmatrix} -1 & 2 \\ -1 & 3 \end{bmatrix}$$

(d)

Let S = $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$ = {s_1, s_2}, B = $\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}$ = {b_1, b_2}

$b_1 = s_1 + s_2$,$b_2 = -s_1 + s_2 \Rightarrow$ $s_1 = 0.5(b_1 - b_2)$, $s_2 = 0.5(b_1 + b_2)$

$[x]_S = \begin{bmatrix}3\\5\end{bmatrix} = 3s_1 + 5s_2 = 1.5(b_1 - b_2) + 2.5(b_1 + b_2) = 4b_1 + b_2$

$\Rightarrow [x]_B = \begin{bmatrix}4\\1\end{bmatrix}$

(e)

$[S]_B$ ：Change basis from S to B(S respect to the basis B)

$[B]_S$ ：Change basis from B to S(B respect to the basis S)

$[T(S)]_S$ ：Linear transform T respect to the basis S

$[T(B)]_B$ ：Linear transform T respect to the basis B

B=S$[B]_S$ (change basis from B to S)

$$\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}[B]_S \quad \Rightarrow \quad [B]_S = \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$$

S= B$[S]_B$ (change basis from S to B)

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}[S]_B \quad \Rightarrow \quad [S]_B = \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}^{-1}$$

$$\begin{array}{cccc} [S] & R^2 \xrightarrow{[T(S)]_S} R^2 & & [S] \\ [B]_S & \uparrow & \downarrow & [S]_B \\ [B] & R^2 \xrightarrow{[T(B)]_B} R^2 & & [B] \end{array}$$

According to the Change-of-coordinate matrix above,

The path $[T(B)]_B$ is equal to $[B]_S \rightarrow [T(S)]_S \rightarrow [S]_B$,which means

$$[T(B)]_B = [S]_B\,[T(S)]_S\,[B]_S = \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}^{-1}\begin{bmatrix}-1 & 2\\-1 & 3\end{bmatrix}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix} = \begin{bmatrix}\frac{3}{2} & \frac{7}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

6.(10pts)

(a)

$$\text{Let S } = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} = \{ v_1, v_2, v_3, v_4 \}$$

$$T(v_1) = Av_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = v_1 + v_3 = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

$$T(v_2) = Av_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = v_2 + v_4 = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

$$T(v_3) = Av_3 = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = v_1 + 2v_3 = \begin{bmatrix} 1 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

$$T(v_4) = Av_4 = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} = v_2 + 2v_4 = \begin{bmatrix} 0 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$[T(S)]_S = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

(b)

$$\text{Let B} = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \right\} = \{ b_1, b_2, b_3, b_4 \}$$

$$T(b_1) = Ab_1 = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} = \frac{5}{2}b_1 + \frac{1}{2}b_3 = \begin{bmatrix} \frac{5}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$T(b_2) = Ab_2 = \begin{bmatrix} -2 & 2 \\ -3 & 3 \end{bmatrix} = \frac{5}{2}b_2 - \frac{1}{2}b_4 = \begin{bmatrix} 0 & \frac{5}{2} & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$T(b_3) = Ab_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}b_1 + \frac{1}{2}b_3 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$T(b_4) = Ab_4 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} = -\frac{1}{2}b_2 + \frac{1}{2}b_4 = \begin{bmatrix} 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$[T(B)]_B = \begin{bmatrix} \frac{5}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{5}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

method 2: As the same procedure in 5.(e)

$$[T(B)]_B = [S]_B \, [T(S)]_S \, [B]_S = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$