Linear Algebra

Problem Set 5 Solution **Spring 2016**

1. (20pts)

(a)

$T: \mathbb{R}^1 \to \mathbb{R}^1, x \epsilon \mathbb{R}^1$, n is a real number.

$$\begin{cases} T(x) = 0x = 0, \text{the graph is a point.} \\ T(x) = nx, \text{the graph is a line pass through the origin.} \end{cases}$$

(b)

$T: \mathbb{R}^2 \to \mathbb{R}^1, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \epsilon \mathbb{R}^2.$

$T(x) = Ax = [a_1 \quad a_2]x.$ a_1, a_2 are real numbers.

$$\begin{cases} T(x) = [0 \quad 0]x = 0, \text{the graph is a point.} \\ T(x) = [a_1 \quad a_2]x = a_1x_1 + a_2x_2, \text{the graph is a plane pass through the origin.} \end{cases}$$

(c)

$x_1, x_2 \in x, T: \mathbb{R}^3 \to \mathbb{R}^1$

$T(x_1) = v \cdot x_1 \quad T(x_2) = v \cdot x_2$

$T(x_1 + ax_2) = v \cdot (x_1 + ax_2) = v \cdot x_1 + v \cdot ax_2 = T(x_1) + aT(x_2) \to$ it is linear.

$T(x) = v \cdot x = v^T x = [3 \quad 2 \quad 4]x$

(d)

$x_1, x_2 \in x, T: \mathbb{R}^3 \to \mathbb{R}^3$

$T(x_1) = x_1 \times y \quad T(x_2) = x_2 \times y$

$T(x_1 + ax_2) = (x_1 + ax_2) \times y = x_1 \times y + a(x_2 \times y) = T(x_1) + aT(x_2) \to$ it is linear.

$$T(x) = x \times y = \begin{bmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{bmatrix} = \begin{bmatrix} 0 & y_3 & -y_2 \\ -y_3 & 0 & y_1 \\ y_2 & -y_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

2.(20pts)

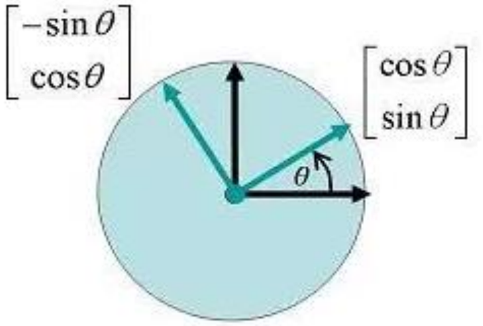
(a)

$$\mathrm{T}\left(\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}\right)=\mathrm{T}\left(\begin{bmatrix}x_1\\0\\\vdots\\0\end{bmatrix}+\begin{bmatrix}0\\x_2\\\vdots\\0\end{bmatrix}+\cdots+\begin{bmatrix}0\\0\\\vdots\\x_n\end{bmatrix}\right)=\mathrm{T}\left(\begin{bmatrix}x_1\\0\\\vdots\\0\end{bmatrix}\right)+\mathrm{T}\left(\begin{bmatrix}0\\x_2\\\vdots\\0\end{bmatrix}\right)+\cdots+\mathrm{T}\left(\begin{bmatrix}0\\0\\\vdots\\x_n\end{bmatrix}\right)$$

$$=x_1\mathrm{T}\left(\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}\right)+x_2\mathrm{T}\left(\begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}\right)+\cdots+x_n\mathrm{T}\left(\begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}\right)$$

(b)

$$\mathrm{T}\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=x_1\mathrm{T}\left(\begin{bmatrix}1\\0\end{bmatrix}\right)+x_2\mathrm{T}\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=x_1\begin{bmatrix}cos\theta\\sin\theta\end{bmatrix}+x_2\begin{bmatrix}-sin\theta\\cos\theta\end{bmatrix}=\begin{bmatrix}cos\theta & -sin\theta\\sin\theta & cos\theta\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$



(c)

$$\mathrm{T}\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\frac{\begin{bmatrix}1\\0\end{bmatrix}^T\begin{bmatrix}3\\1\end{bmatrix}}{\left(\sqrt{10}\right)^2}\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}9/10\\3/10\end{bmatrix},\mathrm{T}\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\frac{\begin{bmatrix}0\\1\end{bmatrix}^T\begin{bmatrix}3\\1\end{bmatrix}}{\left(\sqrt{10}\right)^2}\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}3/10\\1/10\end{bmatrix}$$

$$\mathrm{T}\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=\begin{bmatrix}9/10 & 3/10\\3/10 & 1/10\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

(d)

From 2(c), projection matrix P is $\begin{bmatrix}9/10 & 3/10\\3/10 & 1/10\end{bmatrix}$, and reflection matrix R.

$$Rx=2Px-Ix=\begin{bmatrix}4/5 & 3/5\\3/5 & -4/5\end{bmatrix}x$$

3.(20pts)

(a)True

A is one − to − one, $\ker(A) = \mathrm{N}(A) = \{0\}$, $rank(A) = n$.

(b)True

A is full column rank, that is, A^T is full row rank, $rank(A^T) = \mathrm{n}$. Hence, A^T maps $\mathbb{R}^m$ onto $\mathbb{R}^n$.

(c)True

$\mathrm{C}(A) = \mathbb{R}^m$, $rank(A) = m$.

(d)True

$rank(A) = m = n$, it is nonsigular.

(e)True

A is full column rank, $rank(A) = \mathrm{n}$, A is one − to − one.

(f)True

A is full row rank, $rank(A) = \mathrm{m}$. Hence, A maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.

(g)True

$rank(A) = rank(A^T A) = n$, so $A^T A$ is full rank, it is isomorphism.

(h)False

AA^T is an $m \times m$ matrix, $rank(A) = rank(AA^T) = n$.

AA^T can't be sure full rank and isomorphism.

(i)False

$A^T A$ is an $n \times n$ matrix, $rank(A) = rank(A^T A) = \mathrm{m}$.

$A^T A$ can't be sure full rank and isomorphism.

(j)True

$rank(A) = rank(AA^T) = m$, so AA^T is full rank, it is isomorphism.

4.(25pts)

(a)

$$x = \begin{bmatrix} 3 \\ -1 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightarrow [x]_\beta = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$$

(b)

$$x = \beta[x]_\beta = \beta \mathrm{T}(x)$$

$$\mathrm{T}(x) = \beta^{-1}x = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} x = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} x$$

(c)

$$x = \gamma[x]_\gamma = \beta[x]_\beta$$

$$P = \gamma^{-1}\beta = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

(d)

Let $\beta = \{x_1 , x_2\}$

$$x = \beta[x]_\beta = \beta\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \rightarrow \beta = \mathrm{span}\{\begin{bmatrix} 12 \\ 14 \end{bmatrix}, \begin{bmatrix} -7 \\ -8 \end{bmatrix}\}$$

(e)

From 2(d), we know reflection matrix T is $\begin{bmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{bmatrix}$

Let $\beta = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\mathrm{T}(\beta) = \beta[\mathrm{T}(\beta)]_\beta = T\beta = \begin{bmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{bmatrix}\beta$$

$$[\mathrm{T}(\beta)]_\beta = \beta^{-1}T\beta = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \frac{1}{ad - bc}\begin{bmatrix} \cdots & \frac{3}{5}d^2 + \frac{8}{5}bd - \frac{3}{5}b^2 \\ \frac{3}{5}a^2 - \frac{8}{5}ac - \frac{3}{5}c^2 & \cdots \end{bmatrix}$$

$$\frac{3}{5}d^2 + \frac{8}{5}bd - \frac{3}{5}b^2 = 0 \rightarrow \mathrm{b} = 3, \mathrm{d} = 1$$

$$\frac{3}{5}a^2 - \frac{8}{5}ac - \frac{3}{5}c^2 = 0 \rightarrow \mathrm{a} = 1, \mathrm{c} = -3$$

$$\beta = \mathrm{span}\left\{\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}$$

5.(15pts)

v_1, v_2, v_3 are perpendicular unit vectors in $\mathbb{R}^3$.

(a)

$$T(x) = x - 2(v_3 \cdot x)v_3 = x - 2v_3(v_3 \cdot x) = x - 2v_3{v_3}^T x = (I - 2v_3{v_3}^T)x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} x$$

(b)

$$T(x) = x - 2(v_1 \cdot x)v_2 = x - 2v_2(v_1 \cdot x) = x - 2v_2{v_1}^T x = (I - 2v_2{v_1}^T)x = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x$$

(c)

$v_3 = v_1 \times v_2, v_1 = v_2 \times v_3$

$$T(x) = v_1 \times x + (v_1 \cdot x)v_1 = (v_2 \times v_3) \times x + v_1{v_1}^T x = -x \times (v_2 \times v_3) + v_1{v_1}^T x$$

$$= -[v_2(v_3 \cdot x) - v_3(v_2 \cdot x)] + v_1{v_1}^T x = (v_3{v_2}^T - v_2{v_3}^T + v_1{v_1}^T)x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} x$$