Solution to Problem Set 6

1.

(a)

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ -x_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathrm{Ax}$$

R(T) = C(A) = span{ $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ }

ker(T) = N(A) = span{ $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ }

(b)

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ x_1 + x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathrm{Ax}$$

R(T) = C(A) = span{ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ }

ker(T) = N(A) = span{ $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ }

(c)

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathrm{Ax}$$

R(T) = C(A) = span{ $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ }

ker(T) = N(A) = R^2

(d)

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathrm{Ax}$$

$R(T) = C(A) = \text{span}\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\}$

$\ker(T) = N(A) = \{\begin{bmatrix}0\\1\\0\end{bmatrix}\}$

2.

Assume A is the matrix represented T

$\because$ n = dimV and m = dimW

$\therefore$ $V = \mathbf{R}^n$, W = $\mathbf{R}^m$

→ If $\mathbf{x} \in \mathbf{R}^n$, $y \in \mathbf{R}^m$

→ $A\mathbf{x} = \mathbf{y}$, A is m by n

(a) If T maps $\mathbf{R}^n$ onto $\mathbf{R}^m$

→ All vectors in $\mathbf{R}^m$ will be mapped

→ Range(T) = C(A) = $\mathbf{R}^m$

→ rank(A) = m , A is full row rank

→ $n \geq m$

(b) If T is one-to-one

→ Each vectors in $\mathbf{R}^m$ will be mapped to only one vector in $\mathbf{R}^n$

→ The null space of A must be he zero vector, because only one vector be mapped to zero

→ A is full column rank

→ $m \geq n$

(c) If T is one-to-one and maps $\mathbf{R}^n$ onto $\mathbf{R}^m$

→ According to (a) and (b), the matrix A must satisfy $n \geq m$ and $n \geq m$

→ If m = n , T is one-to-one and maps $\mathbf{R}^n$ onto $\mathbf{R}^m$

3.

(a)

If $\{v_1, v_2, v_3\}$ is a linearly dependent set

Let $v_1 = c_2v_2 + c_3v_3$

$T(v_1) = T(c_2v_2 + c_3v_3) = c_2T(v_2) + c_3T(v_3)$

→$\{ T(v_1) , T(v_2), T(v_3)\}$ is linearly dependent

(b)

$\because \{T(u), T(v)\}$ *is linearly dependent.*

$\rightarrow c_1 T(u) + c_2 T(v) = 0 \ni c_1, c_2 \neq 0$

$\rightarrow T(c_1 u + c_2 v) = 0$

Let $x = c_1 u + c_2 v$

$\because \{u, v\}$ *is linear independent*

$\therefore \ni$ *a nonzero x such that* $T(x) = 0$

4.

(a)

$\because \{1+t, 1+t^2, t+t^2\}$ is a basis for P^2

There will be a set real number of {a, b, c} satisfy that

$p(t) = 6 + 3t - t^2 = a(1+t) + b(1+t^2) + c(t+t^2)$

→ {a, b ,c} = {5, 1, -2}

(b)

The coordinate vector of $1+2t^3$ = [1 0 0 2]

The coordinate vector of $2+t-3t^2$ = [2 1 -3 0]

The coordinate vector of $-t + 2t^2 - t^3$ = [0 -1 2 -1]

Let a matrix be $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 2 & 1 & -3 & 0 \\ 0 & -1 & 2 & -1 \end{bmatrix}$ and the row echelon form is $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

→ The row are independent

∴ $1+2t^3$, $2+t-3t^2$, $-t + 2t^2 - t^3$ are independent

5.

Let a basis V={v_1, v_2, v_3} and another basis W={w_1, w_2, w_3}

Assume T is a linear transformation about mapping the coordinates of V to the coordinates of W.

$\rightarrow A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ where A is the matrix represented T.

$$\rightarrow A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

If $Av = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\rightarrow v = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

6.

(a)

A is the matrix represented T

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow A\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \rightarrow A\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \rightarrow A\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\rightarrow A\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

The matrix A represented T is $\begin{bmatrix} 0.5 & 0.5 & 0.5 \\ -0.5 & 1.5 & 0.5 \\ 0.5 & -0.5 & -0.5 \end{bmatrix}$

(b)

$$Let\, B\; be\, the\;\; basis \{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\}$$

$$A^{'}:[v]_B \xrightarrow{B} v \xrightarrow{A} T(v) \xrightarrow{B^{-1}} [T(v)]_B$$

$$\therefore A^{'} = B^{-1}AB = \begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\\1 & 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}0.5 & 0.5 & 0.5\\-0.5 & 1.5 & 0.5\\0.5 & -0.5 & -0.5\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\\1 & 0 & 1\end{bmatrix} = \begin{bmatrix}0.5 & 0 & -1\\0.5 & 1 & 2\\-0.5 & 0 & 0\end{bmatrix}$$

(c)

$$Let\, B\; be\, the\;\; basis \{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\2\\3\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\}$$

$$A^{'}:[v]_B \xrightarrow{B} v \xrightarrow{A} T(v) \xrightarrow{B^{-1}} [T(v)]_B$$

$$\therefore A^{'} = B^{-1}AB = \begin{bmatrix}1 & 2 & 0\\0 & 2 & 1\\0 & 3 & 1\end{bmatrix}^{-1}\begin{bmatrix}0.5 & 0.5 & 0.5\\-0.5 & 1.5 & 0.5\\0.5 & -0.5 & -0.5\end{bmatrix}\begin{bmatrix}1 & 2 & 0\\0 & 2 & 1\\0 & 3 & 1\end{bmatrix} = \begin{bmatrix}-1.5 & 13.5 & 7\\1 & -5 & -3\\-2.5 & 13.5 & 8\end{bmatrix}$$