2013 spring HW6

1.

(a)

$$[x]_S \xrightarrow{S} x \xrightarrow{B^{-1}} [x]_B \quad , \quad [x]_B = P[x]_S = B^{-1}S[x]_S$$

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$[x]_S \xrightarrow{S} x \xrightarrow{C^{-1}} [x]_B \quad , \quad [x]_C = Q[x]_S = C^{-1}S[x]_S$$

$$Q = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & \frac{3}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}$$

(b)

$$[x]_B \xrightarrow{B} x \xrightarrow{C^{-1}} [x]_C \quad , \quad [x]_C = K[x]_B = C^{-1}B[x]_B$$

$$K = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & -\frac{4}{5} \\ \frac{1}{5} & \frac{3}{5} \end{bmatrix}$$

(c)

$$[T(x)]_S = [T]_S[x]_S$$

$$[T(x)]_B = P[T(x)]_S = P[T]_S[x]_S = [T]_B[x]_B = [T]_B P[x]_S \quad \text{where } P = B^{-1}S$$

$$P[T]_S = [T]_B P \Rightarrow [T]_B = P[T]_S P^{-1} = B^{-1}S[T]_S S^{-1}B = [S]_B[T]_S[B]_S$$

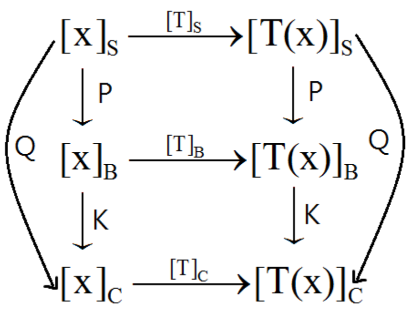
(d)

$$[T(x)]_B = [T]_B[x]_B$$

$$[T(x)]_C = K[T(x)]_B = K[T]_B[x]_B = [T]_C[x]_C = [T]_C K[x]_B \quad \text{where } K = C^{-1}B$$

$$K[T]_B = [T]_C K \Rightarrow [T]_C = K[T]_B K^{-1} = C^{-1}B[T]_B B^{-1}C = [B]_C[T]_B[C]_B$$

(e)



2.

(a)

$$\because S \perp S^{\perp},\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\therefore S^{\perp} = R^3$$

(b)

$$S=C(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\},\ EA = U = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$S^{\perp}=N(A^T) = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

(c)

$$S=C(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\},\ S^{\perp}=N(A^T) = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right\}$$

3.

(a)

$\because C(A^T) \perp N(A)\ \therefore \dim C(A^T) + \dim N(A) = n$

$Ax = A(x_r + x_n)$, $Ax_r \in C(A)$, $x_r \in C(A^T)$, $x_n \in N(A)$

x_r, x_n are linearly independent.

$\dim(x) = \dim(x_r) + \dim(x_n) = \dim C(A^T) + \dim N(A) = n$

$\because$ It is full rank $\therefore x \in R^n$ can be decomposed in a unique way.

(b)

Suppose there exist $x_r' \neq x_r$ such that $Ax_r' = b$, x_r and $x_r \in C(A^T)$

$Ax_r' = b = Ax_r$, $A(x_r - x_r) = 0 \Rightarrow x_r - x_r' \in N(A)$

$\because$ x_r and $x_r \in C(A^T)$ $\therefore$ $x_r - x_r' \in C(A^T)$

$\because C(A^T) \perp N(A)\ \therefore C(A^T) \cap N(A) = \{0\}$

$x_r - x_r' = 0 \Rightarrow x_r = x_r'$ 矛盾

$\Rightarrow$ There is only one and only one vector $x_r \in C(A^T)$ satisfying $Ax_r = b$

4.

(a)

$\because AB = 0 \ \therefore C(B) \in N(A)$

$\Rightarrow \text{rank}(B) = \dim C(B) \le \dim N(A) = 4 - \dim C(A) = 4 - \text{rank}(A)$

$\Rightarrow \text{rank}(B) + \text{rank}(A) \le 4$

(b)No , $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is on $x + y + z = 0$, $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ is on $x - 2y + z = 0$, but $\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \ne 0$

(c)

$$S = C(A^T) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right\},\ S^{\perp} = N(A) = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}\right\}$$

5.

(a)Yes

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \Rightarrow C(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}, a = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P = a(a^T a)^{-1} a^T = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b)

$$p = Pb = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 3/2 \\ 3 \end{bmatrix}$$

(c)

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix} \to R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow N(A) = \text{span}\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}\right\}, N = \begin{bmatrix} -1 & 0 \\ -1 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$Q = N(N^T N)^{-1} N^T = \frac{1}{11}\begin{bmatrix} 5 & 1 & -5 & 2 \\ 1 & 9 & -1 & -4 \\ -5 & -1 & 5 & -2 \\ 2 & -4 & -2 & 3 \end{bmatrix}$$

6.

(a)Yes

$$A^T(b - Ax) = 0 \Rightarrow A^T b = A^T Ax$$

$$x = (A^T A)^{-1} A^T b = \begin{bmatrix} -3/2 \\ 3 \end{bmatrix}$$

(b)Yes

$$x = x_r + x_n = \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix} + a \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5-a \\ a \\ -1 \end{bmatrix}$$

$$\| x \|^2 = \begin{bmatrix} 5-a & a & -1 \end{bmatrix} \begin{bmatrix} 5-a \\ a \\ -1 \end{bmatrix} = 2a^2 - 10a + 26$$

$$\frac{d(2a^2 - 10a + 26)}{da} = 0 \Rightarrow a = \frac{5}{2} \Rightarrow x = \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix} + \frac{5}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 5/2 \\ -1 \end{bmatrix}$$

$$\because Ax = Ax_r + Ax_n = Ax_r \text{ , } x_r \in C(A^T) \;\; \therefore x \in C(A^T)$$