Linear Algebra

Problem Set 6 Solution **Spring 2015**

1.(25pts)

(a)

$$\mathrm{S}[x]_S = x = \mathrm{B}[x]_B$$
$$[x]_B = \mathrm{B}^{-1}\mathrm{S}[x]_S$$
$$[x]_B = \mathrm{P}[x]_S$$

$$\mathrm{P} = \mathrm{B}^{-1}\mathrm{S} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\mathrm{S}[x]_S = x = \mathrm{C}[x]_C$$
$$[x]_C = \mathrm{C}^{-1}\mathrm{S}[x]_S$$
$$[x]_C = \mathrm{Q}[x]_S$$

$$\mathrm{Q} = \mathrm{C}^{-1}\mathrm{S} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$$

(b)

$$\mathrm{C}[x]_C = x = \mathrm{B}[x]_B$$
$$[x]_C = \mathrm{C}^{-1}\mathrm{B}[x]_B$$
$$[x]_C = \mathrm{K}[x]_B$$

$$\mathrm{K} = \mathrm{C}^{-1}\mathrm{B} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$$

(c)

$\mathrm{Q} = [S]_C$: Change basis from S to C(S respect to the basis C)

$\mathrm{Q}^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = [C]_S$: Change basis from C to S(C respect to the basis S)

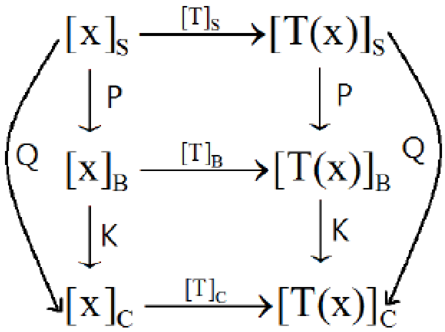
$$[T]_C = [S]_C\,[T]_S\,[C]_S = \mathrm{Q}[T]_S\,\mathrm{Q}^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} [T]_S \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

(d)

$$[B]_C = \mathrm{K} \quad , \quad [C]_B = \mathrm{K}^{-1}$$

$$[T]_B = [C]_B\,[T]_C\,[B]_C = \mathrm{K}^{-1}[T]_C\,\mathrm{K} = \begin{bmatrix} \frac{3}{2} & 2 \\ \frac{1}{2} & 1 \end{bmatrix} [T]_C \begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$$

(e)



2.(15pts)

(a)

$$S = C(R) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\},\ \ S^{\perp} = N(R^T)$$

$$\text{Let } R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix},\ R^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad x_1 = -x_3,\ x_2 = x_3, \text{ let } x_3 = \alpha,\ \ x = \alpha \begin{bmatrix} -1 \\ 1 \\ 1 \\ a \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 0 \\ b \end{bmatrix}$$

$$S^{\perp} = N(R^T) = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ a \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ b \end{bmatrix} \right\} = C(A),\ A = \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 1 & 0 \\ a & b \end{bmatrix}$$

$$S = N(B) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \rightarrow x = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow B = \begin{bmatrix} -1 & 1 & 1 & c \\ 0 & 0 & 0 & d \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 1 & 1 & c \\ 0 & 0 & 0 & d \end{bmatrix}$$

(b)

$N(A) = C(A^T)^{\perp}$

$N(A)^{\perp} = (C(A^T)^{\perp})^{\perp}$

If $A=A^T$ and $C(A)+C(A)^{\perp}=R^n$, $(C(A^T)^{\perp})^{\perp} = C(A)$

for example, $A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c)

NO, $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is on x + y + z = 0, $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ is on x - 2y + z =0, but $\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 1 \neq 0$

3.(10pts)

(a)

$\because C(A^T) \perp N(A)$ $\therefore \dim C(A^T) + \dim N(A)=n$

$Ax = A(x_r + x_n)$, $Ax_r \in C(A)$, $x_r \in C(A^T)$, $x_n \in N(A)$

x_r, x_n are linearly independent

$\dim(x) = \dim(x_r)+\dim(x_n) = \dim C(A^T) + \dim N(A) = n$

$\because$ It is full rank $\therefore x \in R^n$ can be decomposed in a unique way.

(b)

Suppose there exist $x_r' \neq x_r$ such that $Ax_r' = b$, x_r' and $x_r \in C(A^T)$

$Ax_r' = b = Ax_r$, $A(x_r - x_r') = 0 \Rightarrow x_r - x_r' \in N(A)$

$\because$ x_r' and $x_r \in C(A^T)$ $\therefore$ $x_r - x_r' \in C(A^T)$

$\because C(A^T) \perp N(A)$ $\therefore C(A^T) \cap N(A)=\{0\}$

$x_r - x_r' = 0 \Rightarrow x_r = x_r'$ 矛盾

$\Rightarrow$ There is only one and only one vector $x_r \in C(A^T)$ satisfying $Ax_r = b$

(c)

$$\|x\|^2 = \|x_r + x_n\|^2 = (x_r + x_n)^T (x_r + x_n) = (x_r^T + x_n^T)(x_r + x_n)$$

$$= x_r^T x_r + x_r^T x_n + x_n^T x_r + x_n^T x_n$$

If $x_r \in C(A^T)$, $x_r^T x_n = x_n^T x_r = 0$

$$\Rightarrow \|x\|^2 = x_r^T x_r + x_n^T x_n = \|x_r\|^2 + \|x_n\|^2 \text{ is minimized}$$

4.(20pts)

(a)

$$A^T(b-A\hat{x}) = 0$$

$$A^Tb=A^TA\hat{x}$$

$$\hat{x} =(A^TA)^{-1}A^Tb=\begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}=\begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$$

(b)

$$x=\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}+a\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}=\begin{bmatrix} 2 \\ 1+a \\ -a \end{bmatrix}$$

$$\|x\|^2 = 2a^2+2a+5$$

$$\frac{d\|x\|^2}{da}=0 \Rightarrow a = \frac{1}{2} \Rightarrow x=\frac{1}{2}\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$

(c)

$$p = \frac{\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}}{\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}=\begin{bmatrix} \frac{5}{2} \\ \frac{5}{2} \end{bmatrix}$$

$$\hat{x} =\begin{bmatrix} \frac{5}{2} \\ 0 \end{bmatrix}+\alpha\begin{bmatrix} -2 \\ 1 \end{bmatrix}=\begin{bmatrix} \frac{5}{2}-2\alpha \\ \alpha \end{bmatrix}$$

$$\|x\|^2=5\alpha^2-12\alpha+\frac{25}{4}$$

$$\frac{d\|x\|^2}{d\alpha}=0 \Rightarrow \alpha=1 \Rightarrow \hat{x} =\begin{bmatrix} \frac{5}{2} \\ 0 \end{bmatrix}+1\begin{bmatrix} -2 \\ 1 \end{bmatrix}=\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

5.(20pts)

(a)

$$A=\begin{bmatrix}1&0&1\\0&1&0\\1&0&1\\0&1&0\end{bmatrix}\rightarrow\begin{bmatrix}1&0&1\\0&1&0\\0&0&0\\0&0&0\end{bmatrix},\ C(A^T)=\text{span}\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\}$$

$$\text{Let } a=\begin{bmatrix}1&0\\0&1\\1&0\end{bmatrix}$$

$$P=a(a^Ta^{-1})a^T=\begin{bmatrix}\frac{1}{2}&0&\frac{1}{2}\\0&1&0\\\frac{1}{2}&0&\frac{1}{2}\end{bmatrix}$$

(b)

$$Q=I-P=\begin{bmatrix}\frac{1}{2}&0&-\frac{1}{2}\\0&0&0\\-\frac{1}{2}&0&\frac{1}{2}\end{bmatrix}$$

(c)

$$Pb=\begin{bmatrix}\frac{1}{2}&0&\frac{1}{2}\\0&1&0\\\frac{1}{2}&0&\frac{1}{2}\end{bmatrix}\begin{bmatrix}1\\2\\2\end{bmatrix}=\begin{bmatrix}3/2\\2\\3/2\end{bmatrix}$$