Linear Algebra

Problem Set 6 Solution **Spring 2016**

1. (20pts)

(a)

$f = ae^x + be^{-x}, f' = ae^x - be^{-x}$

$[f]_\beta = \begin{bmatrix} a \\ b \end{bmatrix}, [f']_\beta = \begin{bmatrix} a \\ -b \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ -b \end{bmatrix}, \qquad [D]_\beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(b)

$f = ae^x + be^{-x} = \frac{a+b}{2}(e^x + e^{-x}) + \frac{a-b}{2}(e^x - e^{-x})$

$[f]_\gamma = \begin{bmatrix} (a+b)/2 \\ (a-b)/2 \end{bmatrix}$

$[f]_\beta = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} (a+b)/2 \\ (a-b)/2 \end{bmatrix} = P[f]_\gamma,$

$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, Q = P^{-1} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}$

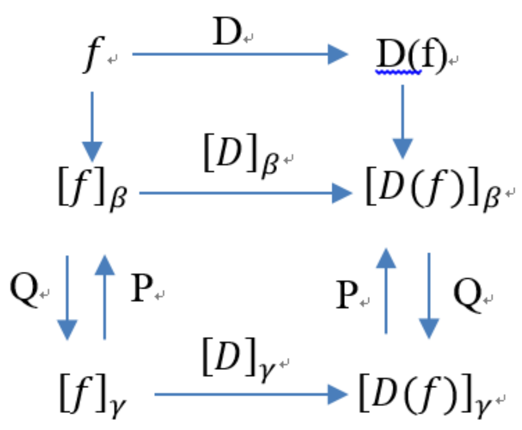
(c)

$[D]_\beta [f]_\beta = [f']_\beta$

$[D]_\beta P[f]_\gamma = P[f']_\gamma = P[D]_\gamma [f]_\gamma$

$[D]_\gamma = Q[D]_\beta P$

(d)



2.(25pts)

(a)

S and T are subspaces of V, so $0 \in S, 0 \in T$.

$0 = 0 + 0 \in S + T$

$x \in S, y \in T$

Let $x_1, x_2 \in x$ and $y_1, y_2 \in y$ and scalar α.

$\alpha x_1 + x_2 \in x$

$\alpha y_1 + y_2 \in y$

$x_1 + y_1 \in S + T$

$x_2 + y_2 \in S + T$

$\alpha(x_1 + y_1) + (x_2 + y_2) = (\alpha x_1 + x_2) + (\alpha y_1 + y_2) \in S + T$

(b)

If $V \in S + T, v = s_1 + t_1, so\ v \in span\{S \cup T\} \rightarrow S + T \subseteq span\{S \cup T\}$

If $V \in span\{S \cup T\}$

$v = s_1 + t_1 + s_2 + t_2 + \cdots + s_n + t_n = (s_1 + s_2 + \cdots + s_n) + (t_1 + t_2 + \cdots + t_n) \in S + T$

$\rightarrow span\{S \cup T\} \subseteq S + T$

Hence, $\mathrm{S} + \mathrm{T} = span\{S \cup T\}$

(c)

$x \in S, y \in S^{\perp}$

Let $x_1, x_2 \in x$ and $y_1, y_2 \in y$

$v = x_1 + y_1 = x_2 + y_2 \rightarrow x_1 - x_2 = y_2 - y_1$

$x_1 - x_2 \in S, y_2 - y_1 \in S^{\perp} \rightarrow (x_1 - x_2)$ and $(y_2 - y_1) \in S \cap S^{\perp} = \{0\}$

So, $x_1 = x_2, y_1 = y_2$, there exists a unique vector x and vector y.

(d)

$S = \mathrm{span}\{x_1, x_2, \cdots x_m\}$

$T = \mathrm{span}\{y_1, y_2, \cdots y_n\}$

Suppose $A = [x_1 \quad \cdots \quad x_m \quad y_1 \quad \cdots \quad y_n]$

$\dim(S + T) = \dim(\mathrm{C}(A)) = rank(A)$

Consider vector c in $\mathrm{N}(A), Ac = 0$

$c_1 x_1 + c_2 x_2 + \cdots + c_m x_m + c_{m+1} y_1 + c_{m+2} y_2 + \cdots + c_{m+n} y_n = 0$

Let $\mathrm{z} = c_1 x_1 + c_2 x_2 + \cdots + c_m x_m = -c_{m+1} y_1 - c_{m+2} y_2 - \cdots - c_{m+n} y_n$

Hence, $\mathrm{z} \in S\ and\ z \in T, z \in S \cap T$

反向推論亦成立，故$S\cap T$為$\mathrm{N}(A), \dim(S\cap T)=\dim \mathrm{N(A)}$
由秩零維數定理得
$\dim \mathrm{N}(A)=\mathrm{m}+\mathrm{n}-\mathrm{rank}(A)\rightarrow \dim(S\cap T)=\dim S+\dim T-\dim(S+T)$

(e)
From (c), (d)
$\dim(S+S^{\perp})=\dim S+\dim S^{\perp}-\dim(S\cap S^{\perp})=\dim S+\dim S^{\perp}=\dim V$

3.(35pts)
(a)

$$S=span\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\1\end{bmatrix}\right\}=C(R), S^{\perp}=N(R^T)$$

$$\text{Let } R=\begin{bmatrix}1&1\\0&0\\1&0\\0&1\end{bmatrix}, R^T=\begin{bmatrix}1&0&1&0\\1&0&0&1\end{bmatrix}\rightarrow\begin{bmatrix}1&0&1&0\\0&0&-1&1\end{bmatrix}$$

$$S^{\perp}=N(R^T)=span\left\{\begin{bmatrix}-1\\0\\1\\1\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right\}$$

(b)

$$S=N(A)=span\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\1\end{bmatrix}\right\}\rightarrow A=\begin{bmatrix}-1&\alpha&1&1\\0&\beta&0&0\end{bmatrix}, \alpha\ and\ \beta \text{ are arbitrary constants.}$$

(c)

$$U=\begin{bmatrix}1&1\\0&0\\1&0\\0&1\end{bmatrix}, V=\begin{bmatrix}-1&0\\0&1\\1&0\\1&0\end{bmatrix}$$

$$P=U(U^TU)^{-1}U^T=\begin{bmatrix}2/3&0&1/3&1/3\\0&0&0&0\\1/3&0&2/3&-1/3\\1/3&0&-1/3&2/3\end{bmatrix}$$

$$Q=V(V^TV)^{-1}V^T=\begin{bmatrix}1/3&0&-1/3&-1/3\\0&1&0&0\\-1/3&0&1/3&1/3\\-1/3&0&1/3&1/3\end{bmatrix}$$

(d)

$$P = \begin{bmatrix} 2/3 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 \\ 1/3 & 0 & 2/3 & -1/3 \\ 1/3 & 0 & -1/3 & 2/3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C(P) = \left\{ \begin{bmatrix} 2/3 \\ 0 \\ 1/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 0 \\ 2/3 \\ -1/3 \end{bmatrix} \right\} = span \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} = S$$

$$N(P) = C(P^T)^{\perp} = C(P)^{\perp} = S^{\perp}$$

(e)

$$v = x + y = a \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{a} = \frac{2}{3}, b = \frac{2}{3}, c = \frac{1}{3}, d = 1$$

$$\text{x} = \begin{bmatrix} 4/3 \\ 0 \\ 2/3 \\ 2/3 \end{bmatrix}, y = \begin{bmatrix} -1/3 \\ 1 \\ 1/3 \\ 1/3 \end{bmatrix}$$

(f)

$C(B) \perp N(B^T)$

$C(B) = (N(B^T))^{\perp} = N(B)^{\perp}$

$\therefore B = B^T$

From(d), $\text{S} = C(P) = N(P)^{\perp}$, and $P^T = P$

$$\text{Hence, let } B = P = \begin{bmatrix} 2/3 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 \\ 1/3 & 0 & 2/3 & -1/3 \\ 1/3 & 0 & -1/3 & 2/3 \end{bmatrix}$$

(g)

$(x - Mx)^T My = x^T My - x^T M^T My = x^T My - x^T My = 0$

Hence, $(x - Mx) \perp My$

4.(20pts)

(a)

$x_n \in N(A)$

$\|x\|^2 = \|x_r + x_n\|^2 = \|x_r\|^2 + \|x_n\|^2 + 2{x_r}^T x_n$

If $x_r \in C(A^T), {x_r}^T x_n = 0$

$\|x\|^2 = \|x_r\|^2 + \|x_n\|^2 \ \geq \|x_r\|^2$

$Ax = A(x_r + x_n) = Ax_r$

When $x = x_r, \|x\|$ is minimized.

(b)

$A^T A\hat{x} = A^T b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \hat{x} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

(c)

When $x_r \in C(A^T)$, we have minimized $\|x\|^2$.

$Ax = A(x_r + x_n) = Ax_r = b$

Let $A^T y = x_r$

$AA^T y = b$

$$x = x_r = A^T(AA^T)^{-1}b = \begin{bmatrix} -7/6 \\ 1/3 \\ 11/6 \end{bmatrix}$$

(d)

$A^T A\hat{x} = A^T b$

$$\begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix} \hat{x} = \begin{bmatrix} 7 \\ 14 \end{bmatrix}$$

We find $\hat{x}$ has infinite solutions.

$C\hat{x} = [5 \quad 10]\hat{x} = 7 = d$

From (c),

$$\hat{x} = C^T(CC^T)^{-1}d = \begin{bmatrix} 7/25 \\ 14/25 \end{bmatrix}$$