Solution to Problem Set 7

1.

(a)

$$B=\{b_1,b_2\}=\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\1\end{bmatrix}\},T(b_1)=\begin{bmatrix}1\\0\\1\end{bmatrix},T(b_2)=\begin{bmatrix}2\\1\\1\end{bmatrix}$$

$$\because [x]_B=\begin{bmatrix}2\\3\end{bmatrix}$$

$$\therefore x=2b_1+3b_2$$

$$\rightarrow T(x)=2T(b_1)+3T(b_2)=\begin{bmatrix}8\\3\\5\end{bmatrix}$$

(b)

$$C=\{c_1,c_2\}=\{\begin{bmatrix}3\\2\\1\end{bmatrix},\begin{bmatrix}2\\-2\\4\end{bmatrix}\}$$

$$b_1=\begin{bmatrix}1\\1\\0\end{bmatrix}=0.4\begin{bmatrix}3\\2\\1\end{bmatrix}-0.1\begin{bmatrix}2\\-2\\4\end{bmatrix}=0.4c_1-0.1c_2$$

$$b_2=\begin{bmatrix}1\\0\\1\end{bmatrix}=0.2\begin{bmatrix}3\\2\\1\end{bmatrix}+0.2\begin{bmatrix}2\\-2\\4\end{bmatrix}=0.2c_1+0.2c_2$$

$$\therefore B_{B\rightarrow C}=\{\begin{bmatrix}0.4\\-0.1\end{bmatrix},\begin{bmatrix}0.2\\0.2\end{bmatrix}\}$$

$$\rightarrow [x]_C=B_{B\rightarrow C}[x]_B=\begin{bmatrix}0.4 & 0.2\\-0.1 & 0.2\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}1.4\\0.4\end{bmatrix}$$

$$T(x)=\begin{bmatrix}8\\3\\5\end{bmatrix}=2.2\begin{bmatrix}3\\2\\1\end{bmatrix}+0.7\begin{bmatrix}2\\-2\\4\end{bmatrix}=2.2c_1+0.7c_2$$

$$\rightarrow [T(x)]_C=\begin{bmatrix}2.2\\0.7\end{bmatrix}$$

(c)

$$c_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 2b_1 + b_2$$

$$c_2 = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = -2b_1 + 4b_2$$

$$\therefore C_{C\to B} = \begin{bmatrix} 2 & -2 \\ 1 & 4 \end{bmatrix}$$

(d)

$$Choose [x_1]_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, [x_2]_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore T(x_1) = T(b_1) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_B$$

$$T(x_2) = T(b_2) = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_B$$

*Let A be matrix representation for T with respect to basis B*

$$A\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\to A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

(e)

$$C = \{c_1, c_2\} = \{\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}\} = \{2b_1 + b_2, -2b_1 + 4b_2\}$$

$$T(c_1) = 2T(b_1) + T(b_2) = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}_C$$

$$T(c_2) = -2T(b_1) + 4T(b_2) = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}_C$$

$$Choose [x_1]_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, [x_2]_C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore T(x_1) = T(c_1) = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}_C$$

$$T(x_2) = T(c_2) = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}_C$$

*Let A be matrix representation for T with respect to basis C*

$$A\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0.5 & 0 \end{bmatrix}$$

$$\to A = \begin{bmatrix} 1 & 2 \\ 0.5 & 0 \end{bmatrix}$$

(f)

$Let\ F\ be\ the\ matrix\ representation\ for\ T\ with\ respect\ to\ basis\ C$

$F = \begin{bmatrix} 1 & 2 \\ 0.5 & 0 \end{bmatrix} \quad \to by\ (e)$

$[T(x)]_C = A[x]_B$

$= F[x]_C$

$= AC_{C\to B}[x]_C$

$\therefore A = FC_{C\to B}{}^{-1} = \begin{bmatrix} 1 & 2 \\ 0.5 & 0 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ 1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 0.2 & 0.6 \\ 0.2 & 0.1 \end{bmatrix}$

2.

$N(A)\ \ contains\ C(B)$

$\to \dim \mathrm{C(B)} \le \dim N(\mathrm{A})$

$\to rank(\mathrm{B}) \le \dim N(\mathrm{A}) \quad ----> (1)$

$rank(A) + \dim N(A) = 4$

$rank(A) = 4 - \dim N(\mathrm{A}) \quad ----> (2)$

$combine (1)\ \&\ (2)$

$\therefore rank(A) + rank(B) \le 4$

3.

(a)

$The\ point\ is\ the\ projection\ of \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} onto\ W$

$\to W(W^T W)^{-1} W^T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.3333 \\ 0.6667 \\ 0.6667 \end{bmatrix}$

(b)

$P_w = W(W^T W)^{-1} W^T = \begin{bmatrix} 0.6667 & 0.3333 & 0.3333 \\ 0.3333 & 0.6667 & -0.3333 \\ 0.3333 & -0.3333 & 0.6667 \end{bmatrix}$

$rank(P_w) = 2$

(c)

$$P_{w^{\perp}} = I - P_w = \begin{bmatrix} 0.3333 & -0.3333 & -0.3333 \\ -0.3333 & 0.3333 & 0.3333 \\ -0.3333 & 0.3333 & 0.3333 \end{bmatrix}$$

$$rank(P_{w^{\perp}}) = 1$$

(d)

$$W^{\perp} = C(P_{W^{\perp}})$$

$$\rightarrow A \;\; basis \;\; for \;\; P_{W^{\perp}} \;\; can \;\; be \begin{bmatrix} 1 \\ \text{-1} \\ \text{-1} \end{bmatrix}$$

4.

$$P \;\; is \;\; an \;\; orthogonal \;\; projection \; matrix \; if \;\; (I-P) \perp P\text{y}$$

$\rightarrow$

$$[(I-P)x]^T P\text{y} = x^T (I-P)^T P\text{y}$$

$$= x^T (I-P)P\text{y} \; ----(\because P^T = P)$$

$$= x^T P\text{y} - x^T P\text{y} ----(\because P^2 = P)$$

$$= 0$$

$$\therefore (I-P)x \perp P\text{y}$$

5.

$$\because p = \frac{a * a^T}{a^T * a}$$

$$\therefore p_1 = \frac{1}{5} * \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} * \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$p_2 = \frac{1}{2} * \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow p_2 * p_1 = \frac{1}{10} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$