2013 spring HW7

1.

(a)

$$u_1 = a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \rightarrow q_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$u_2 = a_2 - u_1\left(\frac{u_1^T a_2}{u_1^T u_1}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \rightarrow q_2 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

$$u_3 = a_3 - u_1\left(\frac{u_1^T a_3}{u_1^T u_1}\right) - u_2\left(\frac{u_2^T a_3}{u_2^T u_2}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \rightarrow q_3 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix},\ Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix},\ R = \begin{bmatrix} a_1^T q_1 & a_2^T q_1 & a_3^T q_1 \\ 0 & a_2^T q_2 & a_3^T q_2 \\ 0 & 0 & a_3^T q_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b)

$$A = QR,\ A^T Ax = Ab \Rightarrow Rx = Q^T b$$

$$x = R^{-1}Q^T b = \frac{1}{2}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}^T \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 0 \\ 0 \end{bmatrix}$$

2.

(a)

$$P = Q(Q^T Q)^{-1}Q^T = QQ^T = q_1 q_1^T + q_2 q_2^T$$

(b)

$$\dim N(Q) = 0,\ N(Q) = \{0\}\ (\because \dim C(Q)=2)$$

$$\dim N(Q^T) = m - n = 3 - 2 = 1$$

$$N(P) = N(QQ^T) = C(Q)^{\perp} = N(Q^T) \Rightarrow \dim N(P) = \dim N(Q^T) = 1$$

(c)

$$u_3 = a - u_1\left(\frac{u_1^T a}{u_1^T u_1}\right) - u_2\left(\frac{u_2^T a}{u_2^T u_2}\right) = a - q_1\left(\frac{q_1^T a}{q_1^T q_1}\right) - q_2\left(\frac{q_2^T a}{q_2^T q_2}\right),\ q_3 = \frac{u_3}{\|u_3\|}$$

(d)

$$Q^TQx = Q^Tb \Rightarrow x = Q^Tb = \begin{bmatrix} {q_1}^T \\ {q_2}^T \end{bmatrix}\left[2q_1 + 3q_2 + 4q_3\right] = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

3.

(a)

$$H^TH = I = H^{-1}H \Rightarrow H^{-1} = H^T$$

$$H^{100} = (HH)^{50} = I^{50} = I$$

(b)

$$Hx = x \ \Rightarrow x \in \text{span}\{u^{\perp}\}$$

$$Hy = -y \ \Rightarrow y \in \text{span}\{u\}$$

(c)

$$\left[I - 2uu^T\right]\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \Rightarrow u = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

4.

(a)False

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A+B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \det(A+B) = 4 \neq 2 = \det(A)+\det(B)$$

(b)False

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \det(2A) = 4 \neq 2 = 2\det(A)$$

(c)True

$$\because \det(AB) = \det(A)\det(B) \ \Rightarrow \det(A^{10}) = \det(A)\det(A)\ldots\det(A) = \det(A)^{10}$$

(d)False

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ A^T = -A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \ \det(A) = 1 \neq 0$$

(e)True

$$A^T = A^{-1} \Rightarrow AA^T = I$$

$$\det(AA^T) = \det(A)\det(A^T) = \det(A)\det(A) = 1 \Rightarrow \det(A) = \pm 1$$

5.

(a)

$$\begin{vmatrix} 0 & A \\ B & C \end{vmatrix} = (-1)^n \begin{vmatrix} B & C \\ 0 & A \end{vmatrix} = (-1)^n \det A \det B$$

(b)

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ -D^{-1}C & I \end{bmatrix} = \begin{bmatrix} A - BD^{-1}C & B \\ 0 & D \end{bmatrix}$$

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} A - BD^{-1}C & B \\ 0 & D \end{vmatrix} = \det(AD - BD^{-1}CD) = \det(AD - BD^{-1}DC) = \det(AD - BC)$$

(c)Yes

$$\begin{vmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \end{vmatrix} \to (-1)^P \begin{vmatrix} * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & * & * \\ * & * & * & * & * \end{vmatrix} \to (-1)^P \begin{vmatrix} * & * & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * \\ * & * & * & * & * \end{vmatrix} = 0$$

6

(a)

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = -1$$

$$\text{Volume} = 1$$

(b)

$$\text{Area} = \det\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \right) = \sqrt{6}$$

(c)

$$T(x,y) = \begin{bmatrix} 4 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \text{Area} = 3^2 \pi \times \begin{vmatrix} 4 & -2 \\ 2 & 3 \end{vmatrix} = 144\pi$$

7.

(a)

$$A=\begin{bmatrix}2&1&1&1\\0&2&1&1\\0&0&2&1\\0&0&0&2\end{bmatrix},\ \det(A)=16,\ A^{-1}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\Rightarrow Ax=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$$x_1=\frac{1}{16}\begin{vmatrix}0&1&1&1\\0&2&1&1\\0&0&2&1\\1&0&0&2\end{vmatrix}=-\frac{1}{16},\ x_2=\frac{1}{16}\begin{vmatrix}2&0&1&1\\0&0&1&1\\0&0&2&1\\0&1&0&2\end{vmatrix}=-\frac{2}{16}$$

$$x_3=\frac{1}{16}\begin{vmatrix}2&1&0&1\\0&2&0&1\\0&0&0&1\\0&0&1&2\end{vmatrix}=-\frac{4}{16},\ x_2=\frac{1}{16}\begin{vmatrix}2&1&1&0\\0&2&1&0\\0&0&2&0\\0&0&0&1\end{vmatrix}=\frac{8}{16}\quad\therefore x=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\frac{-1}{16}\begin{bmatrix}1\\2\\4\\-8\end{bmatrix}$$

(b)

$$A^{-1}=\frac{1}{\det A}C^T\ \Rightarrow C^T=(\det A)A^{-1}$$

$$\det C=\det C^T=\det((\det A)A^{-1})=(\det A)^4(\det A)^{-1}=(\det A)^3=16^3=4096$$

(c)

$$C=((\det A)A^{-1})^T=\begin{bmatrix}8&0&0&0\\-4&8&0&0\\-2&-4&8&0\\-1&-2&-4&8\end{bmatrix}$$

cofactor matrix of C $=((\det C)C^{-1})^T=((\det A)^3\frac{1}{\det A}A^T)^T=((\det A)^2A^T)^T=256\begin{bmatrix}2&1&1&1\\0&2&1&1\\0&0&2&1\\0&0&0&2\end{bmatrix}$