Linear Algebra

Problem Set 7 Solution **Spring 2015**

1.(15pts)

(a)

$$u_1 = a_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, g_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\\0\end{bmatrix}$$

$$u_2 = a_2 - \frac{u_1u_1^T}{u_1^Tu_1}a_2 = \begin{bmatrix}1\\0\\1\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, g_2 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-1\\-1\\3\end{bmatrix}$$

$$u3 = a3 - \frac{u_1u_1^T}{u_1^Tu_1}a_2 - \frac{u_2u_2^T}{u_2^Tu_2}a_3 = \begin{bmatrix}1/3\\-1/3\\-1/3\\1\end{bmatrix}, g3 = \frac{1}{\sqrt{12}}\begin{bmatrix}1\\-1\\-1\\3\end{bmatrix}$$

(b)

$$A=\begin{bmatrix}1&1&1\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}, Q=\begin{bmatrix}1/\sqrt{2}&1/\sqrt{6}&1/\sqrt{12}\\1/\sqrt{2}&-1/\sqrt{6}&-1/\sqrt{12}\\0&2/\sqrt{6}&-1/\sqrt{12}\\0&0&3/\sqrt{12}\end{bmatrix}, R=\begin{bmatrix}\sqrt{2}&1/\sqrt{2}&1/\sqrt{2}\\0&3/\sqrt{6}&1/\sqrt{6}\\0&0&4/\sqrt{12}\end{bmatrix}$$

A = QR

$A^TAx=A^Tb \Rightarrow Rx = Q^Tb$

$$\begin{bmatrix}\sqrt{2}&1/\sqrt{2}&1/\sqrt{2}\\0&3/\sqrt{6}&1/\sqrt{6}\\0&0&4/\sqrt{12}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1/\sqrt{2}&1/\sqrt{6}&1/\sqrt{12}\\1/\sqrt{2}&-1/\sqrt{6}&-1/\sqrt{12}\\0&2/\sqrt{6}&-1/\sqrt{12}\\0&0&3/\sqrt{12}\end{bmatrix}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1/2\\1/2\\1/2\end{bmatrix}$$

2.(20pts)

(a)

$$Q=\begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix}, \text{ project on } C(Q) \Rightarrow P = Q(Q^TQ)^{-1}Q^T = QQ^T$$

$\because C(Q) \perp N(Q^T)$

$\therefore$ project on $N(Q^T)$ $\Rightarrow I - P = I - QQ^T = I - q_1q_1^T - q_2q_2^T$

(b)

dim N(Q) = 0, N(Q) = {0}(dim(Q) = 2)

dim $N(Q^T)$ = m - n = 1

$N(P) = N(QQ^T) = C(Q)^{\perp}$ $\Rightarrow$ dimN(P) = dim $N(Q^T)$ = 1

(c)

$$u3 = a - u_1\left(\frac{u_1^T a}{u_1^T u_1}\right) - u_2\left(\frac{u_2^T a}{u_2^T u_2}\right) = a - q_1\left(\frac{q_1^T a}{q_1^T q_1}\right) - q_2\left(\frac{q_2^T a}{q_2^T q_2}\right) \Rightarrow q_3 = \frac{u_3}{\|u_3\|}$$

(d)

Qx = b

$Q^TQx = Q^Tb$

$$x = Q^Tb = \begin{bmatrix} q_1^T \\ q_2^T \end{bmatrix}(q_1+2q_2+3q_3) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The orthogonal projection of b onto $N(Q^T)$

e = (I - P) b

$=[I - (q_1q_1^T - q_2q_2^T)](q_1+2q_2+3q_3) = 3q_3$

3.(15pts)

(a)

$H^TH = I = H^{-1}H \Rightarrow H^{-1} = H^T$

$H^{99} = H^{100}H^{-1} = (HH)^{50}H^T = I^{50}H^T = H^T$

(b)

Hx = x $\Rightarrow$ $x \in span\{u^{\perp}\}$

Hy = -y $\Rightarrow$ $y \in span\{u\}$

(c)

$$[I - 2uu^T]\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

$$u = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

4.(10pts)

(a) F

Let A = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, det(A) = -2, det(-A) = $(-1)^2$det(A) = det(A)

(b)

T

$det(A^{100}) = detA \cdot derA \cdot derA \cdots derA = det(A)^{100}$

(c)

F

Let A = $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, det(A) = $1 \neq 0$

$A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ = -A

(d)

T

$det(AA^T) = (detA)(detA^T) = det(A)det(A) = 1$, detA = ± 1

(e)

F

Let A = $\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix} \Rightarrow$ A^2= $\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$, det(A) = $0 \neq 1$

5.(10pts)

(a)

$$\begin{vmatrix}0 & A\\ B & C\end{vmatrix} = (-1)^n \begin{vmatrix}B & C\\ 0 & A\end{vmatrix} = (-1)^n \det(B)\det(A)$$

(b)

$$\begin{bmatrix}B & I\\ -AB & 0\end{bmatrix} = \begin{bmatrix}I & 0\\ -A & I\end{bmatrix}\begin{bmatrix}B & I\\ 0 & A\end{bmatrix}$$

$$\begin{vmatrix}B & I\\ -AB & 0\end{vmatrix} = (\det I)^2 \det(A)\det(B) = \det(A)\det(B)$$

6.(10pts)

(a)

$$\overrightarrow{AB} = (0,1,1), \overrightarrow{AC} = (0,1,2), \overrightarrow{AD} = (1,2,3)$$

$$\begin{vmatrix}0 & 1 & 1\\ 0 & 1 & 2\\ 1 & 2 & 3\end{vmatrix} = 1$$

(b)

T(x,y) = $\begin{bmatrix}1 & -2\\ -1 & 5\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} \Rightarrow Area = 5^2 \pi x \begin{vmatrix}1 & -2\\ -1 & 5\end{vmatrix} = 75\pi$

7.(20pts)

(a)

det(A) = 1

$$A^{-1}\begin{bmatrix}0\\0\\1\\0\end{bmatrix}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\Rightarrow Ax=\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$$

$$x_1=\begin{vmatrix}0&0&0&0\\0&1&0&0\\1&2&1&0\\0&3&2&1\end{vmatrix}=0,x_2=\begin{vmatrix}1&0&0&0\\2&0&0&0\\3&1&1&0\\4&0&2&1\end{vmatrix}=0,x_3=\begin{vmatrix}1&0&0&0\\2&1&0&0\\3&2&1&0\\4&3&0&1\end{vmatrix}=1,x_4=\begin{vmatrix}1&0&0&0\\2&1&0&0\\3&2&1&1\\4&3&2&0\end{vmatrix}=-2$$

$$\text{Third column}=\begin{bmatrix}0\\0\\1\\-2\end{bmatrix}$$

(b)

$$A^{-1}=\frac{1}{\det(A)}C^T,\quad C^T=\det(A)A^{-1}\ \Rightarrow\ \det(C^T)=\det(C)=\det(A)^4\det(A)^{-1}=1$$

(c)

$C^T = \det(A)A^{-1}$

$$C^T=\begin{bmatrix}1&0&0&0\\-2&1&0&0\\1&-2&1&0\\0&1&-2&1\end{bmatrix},C=\begin{bmatrix}1&-2&1&0\\0&1&-2&1\\0&0&1&2\\0&0&0&1\end{bmatrix}$$

$$C'=\text{cofactor matrix of }C=\det(C)(C^{-1})^T=\begin{bmatrix}1&0&0&0\\2&1&0&0\\3&2&1&0\\4&3&2&1\end{bmatrix}$$

(d)

$$C^{-1}=\frac{1}{\det C}C'^T=\begin{bmatrix}1&2&3&4\\0&1&2&3\\0&0&1&2\\0&0&0&1\end{bmatrix}$$