Linear Algebra

Problem Set 7 Solution **Spring 2016**

1. (10pts)

$A^T = A^{-1}, B^T = B^{-1}$

(a)False

$(2A)^T(2A) = 4A^TA = 4I$

(b)True

$(-A)^T(-A) = A^TA = I$

(c)True

$(A^2)^T(A^2) = A^{-2}A^2 = I$

(d)True

$(A^{-1})^T(A^{-1}) = I$

(e)True

$(A^T)^T(A^T) = I$

(f)True

$(AB)^T(AB) = B^TA^TAB = I$

(g)False

$(A+B)^T(A+B) = A^TA + A^TB + B^TA + B^TB = 2I + A^TB + B^TA$

(h)True

$(A^{-1}BA)^T(A^{-1}BA) = A^TB^T(A^{-1})^TA^{-1}BA = I$

2.(10pts)

(a)

$$Q^TQ\begin{bmatrix}2/3\\2/3\\1/3\end{bmatrix} = Q^T\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\mathrm{Q} = \begin{bmatrix}1/3 & -2/3 & 2/3\\-2/3 & 1/3 & 2/3\\2/3 & 2/3 & 1/3\end{bmatrix}$$

(b)

$$Q = \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix}, Q \begin{bmatrix} 2 & -3 \\ 3 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 0 & -3 \\ 2 & 0 \end{bmatrix}$$

$$\begin{cases} 2q_1 + 3q_2 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \\ -3q_1 + 2q_2 = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix} \end{cases}, q_1 = \begin{bmatrix} 0 \\ 9/13 \\ 4/13 \end{bmatrix} \to |q_1| \neq 1 \to \text{It doesn't exist orthogonal matrix } Q.$$

3.(30pts)

(a)

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, a_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, a_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, q_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$u_2 = a_2 - \frac{u_1^T a_2}{u_1^T u_1} u_1 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} = q_2$$

$$u_3 = a_3 - \frac{u_1^T a_3}{u_1^T u_1} u_1 - \frac{u_2^T a_3}{u_2^T u_2} u_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix} = q_3$$

(b)

$$A = QR, A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

(c)
$P = A(A^TA)^{-1}A^T = QR(R^TQ^TQR)^{-1}R^TQ^T = QRR^{-1}(R^T)^{-1}R^TQ^T = QQ^T$

$$= \begin{bmatrix} 3/4 & 1/4 & -1/4 & 1/4 \\ 1/4 & 3/4 & 1/4 & -1/4 \\ -1/4 & 1/4 & 3/4 & 1/4 \\ 1/4 & -1/4 & 1/4 & 3/4 \end{bmatrix}$$

(d)

$$Ax = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = b$$

$A^TAx = R^TQ^TQRx = R^TRx = A^Tb = R^TQ^Tb$
$Rx = Q^Tb$

$$x = \begin{bmatrix} -3/4 \\ 3/2 \\ 1/2 \end{bmatrix}$$

(e)
$R\hat{x} = Q^Tb, \hat{x} = 0$
$Q^Tb = 0, b \in \mathrm{N}(Q^T)$

$$b = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}\right\}$$

4.(10pts)
(a)False
$\det(A^T) = \det(-A) = (-1)^n \det A = \det A \to$ when n is even, $\det A$ is an arbitrary number.

(b)True

$$\det(A^T) = \det(A^{-1}) = \frac{1}{\det A} = \det A \to \det A = \pm 1$$

(c)
$\det(A^2) = (\det A)^2 = \det A$
$\to \det A(\det A - 1) = 0 \to \det A = 1 \text{ or } 0$

(d)True
$\det(A^k) = \det A \cdot \det A \cdots \det A = 0 \to \det A = 0$

(e)False

Let $A=\begin{bmatrix}1/2 & 0\\ 0 & 2\end{bmatrix}, A^{-1}=\begin{bmatrix}2 & 0\\ 0 & 1/2\end{bmatrix}$

5.(20pts)

(a)

$$A_n=\begin{bmatrix}1 & 1 & 0 & 0 & \cdots & 0\\ 1 & 2 & 1 & 0 & \cdots & 0\\ 0 & 1 & 2 & 1 & \cdots & 0\\ 0 & 0 & 1 & 2 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \ddots & 1\\ 0 & 0 & 0 & 0 & 1 & 2\end{bmatrix}$$

$\det A_n = 2\det A_{n-1} - \det A_{n-2}$

Let $\mathrm{n}=3, \det A_2 = 1, \det A_1 = 1$

$\det A_3 = 2\det A_2 - \det A_1 = 2(1) - 1 = 1$

$\vdots$

$\det A_n = 2(1) - 1 = 1$

(b)

$$A=\begin{bmatrix}1 & 2 & 8 & 9 & 9\\ 0 & 10 & 7 & 21 & 31\\ 0 & 7 & 5 & 19 & 3\\ 0 & 0 & 0 & 8 & 10\\ 0 & 0 & 0 & 7 & 9\end{bmatrix} \to \begin{bmatrix}1 & 2 & 8 & 9 & 9\\ 0 & 10 & 7 & 21 & 31\\ 0 & 0 & 1/10 & 43/10 & -187/10\\ 0 & 0 & 0 & 8 & 10\\ 0 & 0 & 0 & 0 & 1/4\end{bmatrix}$$

$\det A = 2$

(c)

考慮一個特殊分塊型態 $M=[m_{ij}]=\begin{bmatrix}A & 0\\ 0 & I_{n-r}\end{bmatrix}$，其中 A 是 $r\times r$ 階矩陣。根據排列公式，

$$\det M=\sum_{p}\sigma(p)m_{1j_1}\cdots m_{rj_r}m_{r+1,j_{r+1}}\cdots m_{nj_n}$$
。

因為 M 是分塊對角矩陣且包含 I_{n-r}，

$$m_{r+1,j_{r+1}}\cdots m_{nj_n}=\begin{cases}1 & \text{if } p=(j_1,\ldots,j_r,r+1,\ldots,n)\\ 0 & \text{otherwise.}\end{cases}$$

令 $\tilde{p}=(j_1,\ldots,j_r,r+1,\ldots,n)$ 且 $p_r=(j_1,\ldots,j_r)$。所以，

$$\begin{aligned}\det M&=\sum_{\tilde{p}}\sigma(\tilde{p})m_{1j_1}\cdots m_{rj_r}m_{r+1,j_{r+1}}\cdots m_{nj_n}\\ &=\sum_{p_r}\sigma(p_r)m_{1j_1}\cdots m_{rj_r}\\ &=\det A,\end{aligned}$$

即 $\begin{vmatrix} A & 0 \\ 0 & I \end{vmatrix} = \det A$。同樣道理，$\begin{vmatrix} I & 0 \\ 0 & D \end{vmatrix} = \det D$。

$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$為一上三角矩陣

故 $\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = \begin{vmatrix} A & 0 \\ 0 & D \end{vmatrix} = \begin{vmatrix} A & 0 \\ 0 & I \end{vmatrix}\begin{vmatrix} I & 0 \\ 0 & D \end{vmatrix} = (\det A)(\det D)$

(d)

$$\det\left(\begin{bmatrix} I & 0 \\ -C & A \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \det\left(\begin{bmatrix} I & 0 \\ -C & A \end{bmatrix}\right)\det\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = (\det\mathrm{A})\det\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)$$

$$\det\left(\begin{bmatrix} I & 0 \\ -C & A \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \det\left(\begin{bmatrix} A & B \\ 0 & AD - CB \end{bmatrix}\right) = (\det A)(\det AD - CB)$$

$$\det\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \det(AD - CB)$$

6.(20pts)

(a)

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \rightarrow \sqrt{1+4+1} = \sqrt{6}$$

(b)

$$\frac{1}{2}\left(\begin{vmatrix} 5 & 5 \\ -7 & 7 \end{vmatrix} + \begin{vmatrix} -7 & 7 \\ -5 & -6 \end{vmatrix} + \begin{vmatrix} -5 & -6 \\ 3 & -4 \end{vmatrix} + \begin{vmatrix} 3 & -4 \\ 5 & 5 \end{vmatrix}\right) = 110$$

(c)

$A = \begin{bmatrix} -1 & -6 \\ 1 & 4 \end{bmatrix}$, area of $T(S) = \det A *$ area of $S = 2 * \pi$

(d)

$$T(x,y) = \begin{bmatrix} -x-6y \\ x+4y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} \rightarrow \begin{cases} x = 2u + 3v \\ y = -\frac{1}{2}u - \frac{1}{2}v \end{cases}$$

$$x^2 + y^2 \le 1 \rightarrow (2u+3v)^2 + \left(-\frac{1}{2}u - \frac{1}{2}v\right)^2 \le 1 \rightarrow 17u^2 + 50uv + 37v^2 - 4 \le 0$$

橢圓判別式: $b^2 - 4ac < 0$

$50^2 - 4 * 17 * 37 < 0 \rightarrow$ 為一橢圓