Solution to Problem Set 8

1.

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 7 \\ 1 & 8 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}$$

$$A^T A x = A^T b$$

$$\rightarrow \begin{bmatrix} 4 & 22 \\ 22 & 142 \end{bmatrix} x = \begin{bmatrix} 9 \\ 57 \end{bmatrix}$$

$$\rightarrow x = \begin{bmatrix} 0.2857 \\ 0.3571 \end{bmatrix}$$

$$\rightarrow y = 0.2857 + 0.3571t$$

$$For\ \ y = \beta_0 + \beta_1 t + \beta_2 t^2$$

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 5 & 25 \\ 1 & 7 & 49 \\ 1 & 8 & 64 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}, x = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}$$

$$\therefore Ax = b = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 5 & 25 \\ 1 & 7 & 49 \\ 1 & 8 & 64 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}$$

2.

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, a_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, a_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$u_1 = a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \rightarrow q_1 = \frac{u_1}{\| u_1 \|} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$u_2 = a_2 - \frac{u_1 u_1^T}{u_1^T u_1} a_2 = \begin{bmatrix} -\frac{3}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} \rightarrow q_2 = \frac{u_2}{\| u_2 \|} = \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} \end{bmatrix}$$

$$u_3 = a_3 - \frac{u_1 u_1^T}{u_1^T u_1} a_3 - \frac{u_2 u_2^T}{u_2^T u_2} a_3 = \begin{bmatrix} 0 \\ -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \rightarrow q_3 = \frac{u_3}{\| u_3 \|} = \begin{bmatrix} 0 \\ -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \end{bmatrix}$$

$$A = QR = \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} a_1^T q_1 & a_2^T q_1 & a_3^T q_1 \\ 0 & a_2^T q_2 & a_3^T q_2 \\ 0 & 0 & a_3^T q_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{6} & -\frac{\sqrt{6}}{3} \\ \frac{1}{2} & \frac{\sqrt{3}}{6} & \frac{\sqrt{6}}{6} \\ \frac{1}{2} & \frac{\sqrt{3}}{6} & \frac{\sqrt{6}}{6} \end{bmatrix} \begin{bmatrix} 2 & \frac{3}{2} & 1 \\ 0 & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{3} \\ 0 & 0 & \frac{\sqrt{6}}{3} \end{bmatrix}$$

3.

The projection matrix is $\mathrm{A}(\mathrm{A}^{\mathrm{T}}\mathrm{A})^{-1}\mathrm{A}^{\mathrm{T}}$

But the rank of A *is* 2, $\mathrm{A}^{\mathrm{T}}\mathrm{A}$ *will not be invertible.*

So we remove the dependent columns of A *first and keep the independent columns.*

Then a new matrix $\mathrm{A}=\begin{bmatrix}1 & 1\\ 0 & 1\\ 1 & 1\end{bmatrix}$, *the column space of new* A *and original* A *are the same.*

$\therefore$ *we can use the new* A *to get the projection matrix onto* $C(\mathrm{A})$

$$\mathrm{A}(\mathrm{A}^{\mathrm{T}}\mathrm{A})^{-1}\mathrm{A}^{\mathrm{T}}=\begin{bmatrix}0.5 & 0 & 0.5\\ 0 & 1 & 0\\ 0.5 & 0 & 0.5\end{bmatrix}$$

$\because leftN(A)\perp C(A)$

$$\therefore \textit{The projection matrix onto left } N(A)=I-\mathrm{A}(\mathrm{A}^{\mathrm{T}}\mathrm{A})^{-1}\mathrm{A}^{\mathrm{T}}=\begin{bmatrix}0.5 & 0 & -0.5\\ 0 & 0 & 0\\ -0.5 & 0 & 0.5\end{bmatrix}$$

To find the projection matrix onto row space of A, *we also remove the dependent rows and keep the independent rows, then the new* $A=\begin{bmatrix}1 & 1 & 2 & 1\\ 0 & 1 & 1 & 1\end{bmatrix}$

$$\therefore \textit{The projection matrix onto } C(A^T)=A^T[(A^T)^T A^T]^{-1}(A^T)^T=\begin{bmatrix}0.6 & -0.2 & 0.4 & -0.2\\ -0.2 & 0.4 & 0.2 & 0.4\\ 0.4 & 0.2 & 0.6 & 0.2\\ -0.2 & 0.4 & 0.2 & 0.4\end{bmatrix}$$

And $N(A)\perp C(A^T)$

$$\therefore \textit{The projection matrix onto } N(A)=I-A^T[(A^T)^T A^T]^{-1}(A^T)^T=\begin{bmatrix}0.4 & 0.2 & -0.4 & 0.2\\ 0.2 & 0.6 & -0.2 & -0.4\\ -0.4 & -0.2 & 0.4 & -0.2\\ 0.2 & -0.4 & -0.2 & 0.6\end{bmatrix}$$

4.

(a)

$x_1-x_2+x_3=0$

Let $x_4=c_1, x_3=c_2, x_2=c_3, x_1=c_3-c_2$

$$\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=c_1\begin{bmatrix}0\\0\\0\\1\end{bmatrix}+c_2\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}+c_3\begin{bmatrix}1\\1\\0\\0\end{bmatrix}$$

$$\therefore S=span\{\begin{bmatrix}0\\0\\0\\1\end{bmatrix},\begin{bmatrix}-1\\0\\1\\0\end{bmatrix},\begin{bmatrix}1\\1\\0\\0\end{bmatrix}\}$$

(b)

$$Let\ \ A=\begin{bmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$S^{\perp}=N(A)=span\{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}\}$$

(c)

$$Let\ b_1=c_1\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}+c_2\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}+c_3\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix},b_2=c_4\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$b_1+b_2=c_1\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}+c_2\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}+c_3\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}+c_4\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

$$\rightarrow c_1=-1,c_2=-\frac{2}{3},c_3=\frac{2}{3},c_4=-\frac{1}{3}$$

$$\therefore b_1=\begin{bmatrix} \frac{4}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \\ -1 \end{bmatrix},b_2=\begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \\ 0 \end{bmatrix}$$

5.

(a)

$Qu=(I-2uu^T)u$

$=u-2uu^Tu$

$=-u$

(b)

$Qv=(I-2uu^T)v$

$=v-2uu^Tv$

$=v$

6.

R *is an triangular matrix, and* $A = Q_1 R$

$\therefore C(A) = C(Q_1)$, Q_1 *is the column space of* A

$Q_2{}^T A = Q_2{}^T Q_1 R = 0 \quad \rightarrow Q$ *is orthogonal matrix* $\therefore Q_2{}^T Q_1 = 0$

$\therefore Q_2$ *is left null space of* A