2013 spring HW8

1.

$$A=\begin{bmatrix}1&2&3\\1&2&3\\1&2&3\end{bmatrix}\rightarrow\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$Ax=0=0\times x\Rightarrow \lambda_1=\lambda_2=0$

$tr(A)=\sum_{i=1}^{3}\lambda_i=6\Rightarrow\lambda_3=6$

when $\lambda_1=\lambda_2=0$, $(A-0I)x=0$, eigenvector $=\text{span}\left\{\begin{bmatrix}-2\\1\\0\end{bmatrix}\begin{bmatrix}-3\\0\\1\end{bmatrix}\right\}$

when $\lambda_3=6$, $(A-6I)x=0$, eigenvector $=\text{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$

A is diagonalizable because A has linearly independent eigenvectors.

2.

(a) $\because$ A doesn't have zeor eigenvalue, nullity(A)=0 $\therefore$ rank(A)=3

(b) $\because \det(A-\lambda I)=\det(A-\lambda I)^T=\det(A^T-\lambda I)=0$ $\therefore$ eigenvalues is still 1,1,2

(c) $\because A=S\Lambda S^{-1}, A^3=S\Lambda^3S^{-1}$ $\therefore$ eigenvalues is 1,1,8

(d) $\because A=S\Lambda S^{-1}\Rightarrow A^2=S\Lambda^2S^{-1}\Rightarrow A^2+I=S(\Lambda^2+I)S^{-1}\Rightarrow(A^2+I)^{-1}=S(\Lambda^2+I)^{-1}S^{-1}$

$\therefore$ eigenvalues is 1/2,1/2,1/5

(e) $\det(A^TA)=\det(A^T)\det(A)=\det(A)\det(A)=(\lambda_1\lambda_2\lambda_3)(\lambda_1\lambda_2\lambda_3)=4$

(f) $\because A+2I=S(\Lambda+2I)S^{-1}$ $\therefore \det(A+2I)=3\times3\times4=36$

(g) $\because A+A^{-1}=S(\Lambda+\Lambda^{-1})S^{-1}$ $\therefore \det(A+A^{-1})=(1+1/1)\times(1+1/1)\times(2+1/2)=10$

(h) $tr(A+2I)=tr(A)+tr(2I)=(1+1+2)+(2+2+2)=10$

(i) $tr(3A+4A^T)=tr(3A)+tr(4A^T)=tr(3A)+tr(4A)=7tr(A)=7\times4=28$

(j) $tr(PAP^{-1})=tr(A)=4$

3.

(a) $\because$ The eigenvalues of a triangular matrix are the diagonal entries.

$\therefore$ eigenvalues are 5, 1, 3, 2

(b) $\because$ eigenvalue of $\begin{bmatrix}4&0\\-2&5\end{bmatrix}$ are 4 , 5 ; eigenvalue of $\begin{bmatrix}3&3\\-4&-5\end{bmatrix}$ are -3 , 1;

$\therefore$ eigenvalues are 4, 5, 1, -3

(c) $C=\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}\to\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$

$Ax=0=0\times x\Rightarrow \lambda_1=\lambda_2=\lambda_3=0$, $tr(A)=\sum_{i=1}^{4}\lambda_i=4\Rightarrow\lambda_4=4$

4.

(a) True

$\because$ Suppose A has an eigenvalue 0, then $\det(A)=\lambda_1\lambda_2......\lambda_n=0$,which means A is not invertible.

$\therefore$ If A is invertible, then 0 is not an eigenvalue of A.

(b)True

$\det(A^2)=\det(A)\det(A)=0\Rightarrow\det(A)=\lambda_1\lambda_2......\lambda_n=0\Rightarrow$ A is not invertible.

$\Rightarrow$ eigenvalues of A are the same $\Rightarrow\lambda_1=\lambda_2=......=\lambda_n=0$

(c)True

$\because$ different eigenvalue $\therefore$ eigenvectors is linearly independent

(d)True

Suppose x_1 and x_2 are eigenvector of A , $Ax_1=0$, $Ax_2=0$

$A(x_1+x_2)=Ax_1+Ax_2=0=0$, x_1+x_2 are also an eigenvector of A.

(e)False

$N((A-\lambda I)^T)=N(A^T-\lambda I)\neq N(A-\lambda I)$

(f)True

$\because$ A has distinct eigenvalue $\therefore$ eigenvectors are linearly independent.

$\Rightarrow$ A is diagonalizable.

(g)False

$A=\begin{bmatrix}1&2\\0&1\end{bmatrix}\to\begin{bmatrix}1&0\\0&1\end{bmatrix}=I$, but A is not diagonalizable since eigenvectors $x_1=x_2=\begin{bmatrix}1\\0\end{bmatrix}$

(h)False

$A=I$ is diagonalizable.

(i)True

A is diagonlaizable , $A=S\Lambda S^{-1}$

$A^T=(S\Lambda S^{-1})^T=(S^{-1})^T\Lambda^T S^T=(S^T)^{-1}\Lambda S^T\Rightarrow A^T$ is diagonlaizable

(j)False

$\because$ A is diagonlaizable $\therefore$ A has distinct eigenvalues.

But eigenvalue of A could be $0\Rightarrow rank(A)\neq n$

5.

(a)

$$A=\begin{bmatrix}4 & -3\\ 2 & -1\end{bmatrix},\ \lambda_1=2,\ \lambda_2=1,\ x_1=\begin{bmatrix}3\\ 2\end{bmatrix},\ x_2=\begin{bmatrix}1\\ 1\end{bmatrix},\ S=\begin{bmatrix}3 & 1\\ 2 & 1\end{bmatrix}$$

$$A^8=S\Lambda^8S^{-1}=\begin{bmatrix}3 & 1\\ 2 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}^8\begin{bmatrix}3 & 1\\ 2 & 1\end{bmatrix}^{-1}=\begin{bmatrix}766 & -765\\ 510 & -509\end{bmatrix}$$

(b)

$$A=SDS^{-1}\Rightarrow D=S^{-1}AS=\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}^{-1}\begin{bmatrix}-3 & 12\\ -2 & 7\end{bmatrix}\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix}$$

6.

(a)

$$(A-\lambda I)=S\Lambda S^{-1}-\lambda I=S(\Lambda-\lambda I)S^{-1}$$

$$(A-\lambda_1 I)(A-\lambda_2 I)\ldots\ldots(A-\lambda_n I)=S(\Lambda-\lambda_1 I)S^{-1}S(\Lambda-\lambda_2 I)S^{-1}\ldots\ldots S(\Lambda-\lambda_n I)S^{-1}$$

$$=S\begin{bmatrix}0 & & & \\ & \lambda_2-\lambda_1 & & \\ & & \ddots & \\ & & & \lambda_n-\lambda_1\end{bmatrix}\begin{bmatrix}\lambda_1-\lambda_2 & & & \\ & 0 & & \\ & & \ddots & \\ & & & \lambda_n-\lambda_2\end{bmatrix}\begin{bmatrix}\lambda_1-\lambda_n & & & \\ & \lambda_2-\lambda_n & & \\ & & \ddots & \\ & & & 0\end{bmatrix}S^{-1}=0$$

Then $p(A)=$ zero matrix which is the Cayley-Hamilton Theorem.

(If A is not diagonalizable, one proof is to take a sequence of diagonalizable matrices approaching A)

(b)

$\lambda=1$, -1 , -3

$$(A-I)(A+I)(A-3I)=A^3-3A^2-A+3I=0$$

$$A^3-3A^2-A=-3I$$

$$A^2-3A-I=-3A^{-1}$$

$$A^{-1}=\frac{-1}{3}(A^2-3A-I)=\begin{bmatrix}-1/3 & 2/3 & 0\\ 2/3 & -1/3 & 0\\ 0 & 0 & 1\end{bmatrix}$$