Linear Algebra

Problem Set 8 Solution **Spring 2015**

1.(15pts)

A = $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$, det(A) = 0, tr(A) = 1+4+1 = 6.

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Rank-one matrix} \Rightarrow \lambda_1 = \lambda_2 = 0$$

tr(A) = 6 = $\lambda_1 + \lambda_2 + \lambda_3 \Rightarrow \lambda_3$=6

$\Rightarrow \lambda = 0, 0, 6$

$$Ax_3 = \lambda_3 x_3 \Rightarrow (A - \lambda_3 I)x_3 = 0 \Rightarrow \begin{bmatrix} -5 & 2 & 1 \\ 2 & -2 & 2 \\ 1 & 2 & -5 \end{bmatrix} x_3 = 0$$

$x_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ (By finding Nullspace of $A - \lambda_3 I$)

$$Ax_1 = \lambda_1 x_1 \Rightarrow Ax_1 = 0 \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} x_1 = 0,$$

$$x_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

For $\lambda_1 = \lambda_2 = 0$, eigenvector = $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

For $\lambda_3 = 6$, eigenvector = $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$Ax = \lambda x \quad \Rightarrow \quad (A+3I)x = Ax + 3x = \lambda x + 3x = (\lambda+3)x$

$(A+3I)x = (\lambda+3)x \Rightarrow$ Eigenvalue of $(A+3I): \lambda' = \lambda + 3 = 3,\ 3,\ 9$

Eigenvectors x' = x $\Rightarrow$

For $\lambda_1' = \lambda_2' = 3$, eigenvector = $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

For $\lambda_3' = 9$, eigenvector = $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$Ax = \lambda x \quad \Rightarrow \quad A^9 Ax = A^9 \lambda x \Rightarrow A^{10}x \ = (\lambda A^9 x) \ = (\lambda^{10} x)$

$A^{10}x = \lambda^{10} x \Rightarrow$ Eigenvalue of $(A^{10}): \lambda' = \lambda^{10} = 0,\ 0,\ 6^{10}$

Eigenvectors x' = x

For $\lambda_1' = \lambda_2' = 0$, eigenvector = $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

For $\lambda_3' = 6^{10}$, eigenvector = $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

2.(15pts)

(a)

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \text{ rank-one matrix} \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$$

$$\text{tr}(A) = 1 + 1 + 1 + 1 = 4 \Rightarrow \lambda_4 = 4$$

$$\Rightarrow \lambda = 0, 0, 0, 4$$

(b)

$$B = A - I \Rightarrow \lambda' = \lambda - 1 \Rightarrow \lambda' = -1, -1, -1, 3$$

(c)

Method 1 Ref.(每週問題 June 4, 2012)

$$D = C - I = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \Rightarrow D = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \lambda' \begin{bmatrix} u \\ v \end{bmatrix} \Rightarrow I \times v = \lambda' u, \quad I \times u = \lambda' v$$

$$\Rightarrow u = \lambda'^2 u \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u = \lambda'^2 u \Rightarrow \quad \lambda'^2 = 1, 1 \Rightarrow \lambda' = \pm 1, \pm 1$$

$$\Rightarrow D = C - I \Rightarrow \lambda = \lambda' + 1 = 0, 0, 2, 2$$

Method 2

$$\text{Let} \quad \det(C - \lambda I) = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 & 1 & 0 \\ 0 & 1-\lambda & 0 & 1 \\ 1 & 0 & 1-\lambda & 0 \\ 0 & 1 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[(1-\lambda)^3 - (1-\lambda)] + [1 - (1-\lambda)^2] = 0 \Rightarrow \lambda = 0, 0, 2, 2$$

3.(20pts)

(a)

$\lambda' = 1, 2, 0$,nullity = 1, rank(A+I) = 2

(b)

$A^T x = \lambda' x \quad \Rightarrow$ Let $\det(A^T - \lambda' I) = 0$

$Ax = \lambda x \quad \Rightarrow$ Let $\det(A - \lambda I) = 0$

$\Rightarrow \det(A^T - \lambda' I) = \det(A - \lambda I) \Rightarrow \lambda' = \lambda$

$A^T Ax = A^T \lambda x = \lambda A^T x = \lambda\lambda' x = \lambda^2 x$

Eigenvalue of ($A^T A$) = λ^2 = 0, 1, 1

(c)

$Ax = \lambda x \Rightarrow A^{-1}Ax = A^{-1}\lambda x \Rightarrow Ix = \lambda A^{-1}x \Rightarrow \frac{1}{\lambda}x = A^{-1}x$

$\Rightarrow$Eigenvalue of $(A^{-1}) = \frac{1}{\lambda}$

$\Rightarrow$Eigenvalue of $(A + 2I)^{-1}$: $\lambda' = \frac{1}{\lambda + 2}$ =1, $\frac{1}{2}$, $\frac{1}{3}$

(d)

Eigenvalue of $(A^2) = \lambda^2$ =0, 1, 1

Eigenvalue of ($A^2 + I$)= λ^2 +1=1, 2, 2

Eigenvalue of ($A^2 + I$) ^{2} =(λ^2 +1) ^{2} =1, 4, 4

(e)

det(A) = product of all eigenvalue

Eigenvalue of $3A - 2I$ =-2, 1, -5

det($3A - 2I$) = 10

(f)

$Ax = \lambda x$, $Bx = \lambda_B x$, $B^{-1}x = \frac{1}{\lambda_B}x$

$\Rightarrow ABx = A\lambda_B x = \lambda_B Ax = \lambda_B \lambda x$

$\Rightarrow B^{-1}ABx = B^{-1}\lambda\lambda_B x = \lambda\lambda_B B^{-1}x = \lambda\lambda_B \frac{1}{\lambda_B}x = \lambda x$

Eigenvalue of $(B^{-1}AB) = \lambda \Rightarrow$ det($B^{-1}AB$) = $0 \times -1 \times 1 = 0$

(g)

Eigenvalue of $(A^2 + A) = \lambda^2 + \lambda = 0, 0, 2 \Rightarrow$ det($A^2 + A$) = 0

(h)

Eigenvalue of (A+2I) = $\lambda + 2 = 2$,3 ,1

tr(A) = sum of eigenvalue = 1 + 2 + 3 = 6

(i)

Eigenvalue of (3A+4A^T) = $3\lambda + 4\lambda = 7\lambda = 0$, 7, -7

tr(A) = sum of eigenvalue = 0

(j)

Eigenvalue of $(B^{-1}AB) = \lambda$

tr(A) = sum of eigenvalue = 0

4.(20pts)

(a)

True

det(A) = product of eigenvalue, if eigenvalue = 0, det(A) = 0 $\Rightarrow$ singular

(b)

True

det(A^2) = det(A)det(A) = 0 $\Rightarrow$ det(A) = $\lambda_1\lambda_2......\lambda_n = 0 \Rightarrow$ A is not invertible

$\Rightarrow$ eigenvalues of A are the same $\Rightarrow \lambda_1 = \lambda_2 = \lambda_3 ...... = \lambda_n = 0$

(c)

True

$\because$ different eigenvalue $\therefore$ eigenvectors are linearly independent

(d)

False

Let A = $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $\lambda = 0, 0, 0$

(e)

False

N($A^T - \lambda I$) $\neq$ N($A - \lambda I$)

(f)

True

$\because$ distinct eigenvalue $\therefore$ eigenvectors are linearly independent

$\Rightarrow$ A is diagonalizable

(g)

False

counterexample: $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \lambda = 0, 0, 1$, rank(A) = 2

(h)

True

nullity = 1, rank(A) = 2

(i)

False

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is diagonalizable

(j)

True

A is diagonalizable A= $S\Lambda S^{-1}$

$A^T = (S\Lambda S^{-1})^T = (S^{-1})^T \Lambda^T S^T = (S^T)^{-1} \Lambda S^T \Rightarrow$ A^T is diagonalizable

5.(20pts)

(a)

Let $\det(A-\lambda I)=0 \Rightarrow \lambda=1, 3$

$\lambda=1 \Rightarrow x_1=\begin{bmatrix}1\\1\end{bmatrix}, \lambda=3 \Rightarrow x_1=\begin{bmatrix}-1\\1\end{bmatrix}$

Let S = $[\,x_1 \quad x_2\,] = \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$

AS $=[\,Ax_1 \quad Ax_2\,] = [1x_1 \quad 3x_2\,] = \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 3\end{bmatrix} = S\Lambda$

$S^{-1}AS=\Lambda=\begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}$

$A^k = S\Lambda^k S^{-1} = \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}^k\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}^{-1}$

(b)

Let $\det(R-\lambda I) = (\cos\theta-\lambda)^2+\sin^2\theta = 0$

$\lambda = \cos\theta \pm i\sin\theta$

$\lambda_1 = \cos\theta + i\sin\theta,\quad x_1=\begin{bmatrix}1\\-i\end{bmatrix},\quad \lambda_2 = \cos\theta - i\sin\theta,\quad x_2=\begin{bmatrix}1\\i\end{bmatrix}$

Let S = $[\,x_1 \quad x_2\,] = \begin{bmatrix}1 & 1\\-i & i\end{bmatrix}$

RS $=[\,Rx_1 \quad Rx_2\,] = [\,\lambda_1 x_1 \quad \lambda_2 x_2\,] = \begin{bmatrix}1 & 1\\-i & i\end{bmatrix}\begin{bmatrix}\cos\theta+i\sin\theta & 0\\0 & \cos\theta-i\sin\theta\end{bmatrix} = S\Lambda$

$S^{-1}RS=\Lambda=\begin{bmatrix}\cos\theta+i\sin\theta & 0\\0 & \cos\theta-i\sin\theta\end{bmatrix}$

$R^n = \begin{bmatrix}1 & 1\\-i & i\end{bmatrix}\begin{bmatrix}e^{i\theta} & 0\\0 & e^{-i\theta}\end{bmatrix}^n\begin{bmatrix}1 & 1\\-i & i\end{bmatrix}^{-1} = \frac{1}{2i}\begin{bmatrix}i(e^{ni\theta}+e^{-ni\theta}) & -(e^{ni\theta}-e^{-ni\theta})\\(e^{ni\theta}-e^{-ni\theta}) & i(e^{ni\theta}+e^{-ni\theta})\end{bmatrix}$

$=\frac{1}{2i}\begin{bmatrix}2i\cos n\theta & -2i\sin n\theta\\2i\sin n\theta & 2i\cos n\theta\end{bmatrix} = \begin{bmatrix}\cos n\theta & -\sin n\theta\\\sin n\theta & \cos n\theta\end{bmatrix}$

(c)

$(A-\lambda I) = S(\Lambda-\lambda I)S^{-1}$

$(A-\lambda_1 I)(A-\lambda_2 I)\ ...\ (A-\lambda_n I) = S(\Lambda-\lambda_1 I)S^{-1}\ S(\Lambda-\lambda_2 I)S^{-1} ...\ S(\Lambda-\lambda_n I)S^{-1}$

$$=S\begin{bmatrix} 0 & & & \\ & \lambda_2-x_1 & & \\ & & \ddots & \\ & & & \lambda_n-x_1 \end{bmatrix}\begin{bmatrix} \lambda_1-x_2 & & & \\ & 0 & & \\ & & \ddots & \\ & & & \lambda_n-x_2 \end{bmatrix}\cdots\begin{bmatrix} \lambda_1-x_n & & & \\ & \lambda_2-x_n & & \\ & & \ddots & \\ & & & \lambda_n-x_n \end{bmatrix}S^{-1}=0$$

$\Rightarrow$ p(A) = zero matrix which is the Cayley-Hamilton theorem

(If A is not diagonalizable, one proof is to take a sequence of diagonalizable matrices approaching A)

(d)

Let det($A-\lambda I$)=0$\Rightarrow \lambda = 1, 1, 3$

(A-I)(A-I)(A-3I) = 0

$\Rightarrow A^3 - 5A^2 + 7A = 3I$

$\Rightarrow A^2 - 5A + 7I = 3A^{-1}$

$$A^{-1} = \frac{1}{3}(A^2 - 5A + 7I) = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6.(10pts)

(a)

$$\begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}\begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$$

$G_{k+2} = \frac{1}{2}G_{k+1} + \frac{1}{2}G_k$

$G_{k+1} = G_{k+1}$

Let det($A-\lambda I$) = 0$\Rightarrow$(2λ+1)(λ-1) =0$\Rightarrow \lambda = 1, -\frac{1}{2}$

$\lambda = 1 \Rightarrow x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \lambda = -\frac{1}{2} \Rightarrow x_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Let S = $[\,x_1 \quad x_2\,] = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$

$$\text{AS} = [\,Ax_1 \quad Ax_2\,] = [\,1x_1 \quad -\frac{1}{2}x_2\,] = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix} = S\Lambda$$

$$S^{-1}AS = \Lambda = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}^{-1}$$

$$\lim_{k\to\infty} A^n = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \lim\limits_{k\to\infty} 1^n & 0 \\ 0 & \lim\limits_{k\to\infty}(-\frac{1}{2})^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 2/3 & 1/2 \\ 2/3 & 1/3 \end{bmatrix}$$

(b)

$$\lim_{k\to\infty} \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \lim_{k\to\infty} A^n \begin{bmatrix} G_1 \\ G_0 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/2 \\ 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix}$$