Linear Algebra

Problem Set 8 Solution **Spring 2016**

1. (10pts)

(a)

$\det(A - \lambda \mathrm{I}) = (4 - \lambda)(6 - \lambda) = 0$

$\lambda_1 = 4, \begin{bmatrix} 0 & 5 \\ 0 & 2 \end{bmatrix} x_1 = 0, x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_2 = 6, \begin{bmatrix} -2 & 5 \\ 0 & 0 \end{bmatrix} x_2 = 0, x_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

(b)

$\det(A - \lambda \mathrm{I}) = (1 - \lambda)(1 - \lambda) - 1 = 0$

$\lambda_1 = 0, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x_1 = 0, x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 2, \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} x_2 = 0, x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(c)

$\det(A - \lambda \mathrm{I}) = (1 - \lambda)(1 - \lambda) = 0$

$\lambda = 1, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x = 0, x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

2.(15pts)

(a)

$A^2 x = A\lambda x = \lambda^2 x$

Yes, λ^2

(b)

$A^{-1}Ax = A^{-1}\lambda x \rightarrow A^{-1}x = \lambda^{-1}x$

Yes, λ^{-1}

(c)

$(A + 2I)x = Ax + 2x = (\lambda + 2)x$

Yes, $\lambda + 2$

(d)
$5Ax = 5\lambda x$
Yes, 5λ

(e)
$(A^2 + 2A + 3I)x = (\lambda^2 + 2\lambda + 3)x$
Yes, $\lambda^2 + 2\lambda + 3$

3.(20pts)
(a)True
$(A - \lambda \mathrm{I})x = Bx = 0,$
$\dim N(\mathrm{B}) = \mathrm{n} - \mathrm{rank}(B)$
$\because \det(\mathrm{B}) = 0 \rightarrow \mathrm{rank}(B) < \mathrm{n}$
$\dim N(\mathrm{B}) = \dim N(A - \lambda \mathrm{I}) \neq 0$
$\therefore N(A - \lambda \mathrm{I}) \neq \{0\}$

(b)True
$\det(A - \lambda \mathrm{I}) = \det(A - \lambda \mathrm{I})^T = \det(A^T - \lambda \mathrm{I}) = 0$

(c)False
$N(A - \lambda \mathrm{I}) \neq N(A^T - \lambda \mathrm{I})$

(d)True
$Ax = \lambda_1 x, Bx = \lambda_2 x$
$(A + B)x = (\lambda_1 + \lambda_2)x$

(e)True
From (d)
$ABx = \lambda_2 Ax = \lambda_1 \lambda_2 x$

(f)True

$$\det(A) = \prod_{i=1}^{n} \lambda_i = 0$$

(g)True
$A^2 x = \lambda^2 x = 0$
$\because x \neq 0 \rightarrow \lambda = 0$

(h)True

$Ax = \lambda x, \lambda = 0,1,2$

When $\lambda = 0$

$\dim N(A - \lambda \text{I}) = \dim N(A) = 1$

$\therefore \text{rank}(A) = 2$

(i)True

A is not diagonalizable, and $\lambda = 0,0,1$

When $\lambda = 0$

$\dim N(A - \lambda \text{I}) = \dim N(A) = 1$

$\therefore \text{rank}(A) = 2$

(j)True

When $\lambda = 0$

$\dim N(A - \lambda \text{I}) = \dim N(A) = 1$

$\therefore \text{rank}(A) = 2$

4.(15pts)

(a)

$$u = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix}$$

$$R = \begin{bmatrix}3/5 & -4/5\\-4/5 & -3/5\end{bmatrix}$$

$\lambda = 1, -1$

$$x = \begin{bmatrix}-2\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}$$

(b)

$$R(\theta) = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

$$R(180^{\circ}) = \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$$

$\lambda = -1, -1$

$$x = \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$$

(c)

$u = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$P = \dfrac{uu^T}{u^T u} = \dfrac{1}{13}\begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}$

$\lambda = 0,1$

$x = \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

5.(20pts)

$\lambda = 1, -1/4 \quad x = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(a)

$A = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1/4 \end{bmatrix} \begin{bmatrix} 1/5 & 1/5 \\ 2/5 & -3/5 \end{bmatrix}$

(b)

$A^k = S\Lambda^k S^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (-1/4)^k \end{bmatrix} \begin{bmatrix} 1/5 & 1/5 \\ 2/5 & -3/5 \end{bmatrix}$

(c)

$\lim_{k\to\infty} A^k = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/5 & 1/5 \\ 2/5 & -3/5 \end{bmatrix} = \begin{bmatrix} 3/5 & 3/5 \\ 2/5 & 2/5 \end{bmatrix}$

(d)True

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$A^k x = \begin{bmatrix} 3/5 & 3/5 \\ 2/5 & 2/5 \end{bmatrix} x = \begin{bmatrix} \frac{3}{5}x_1 + \frac{3}{5}x_2 \\ \frac{2}{5}x_1 + \frac{2}{5}x_2 \end{bmatrix} = \frac{1}{5}(x_1 + x_2)\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

6.(10pts)

(a)

$(A - \lambda_1 I)(A - \lambda_2 I) \cdots (A - \lambda_n I) = S(\Lambda - \lambda_1 I)(\Lambda - \lambda_2 I) \cdots (\Lambda - \lambda_n I) S^{-1}$

$= S \begin{bmatrix} p(\lambda_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & p(\lambda_n) \end{bmatrix} S^{-1} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$

(b)

$\det(A - \lambda \mathrm{I}) = -\lambda^3 + 6\lambda^2 - 7\lambda - 2 = 0$

$-A^3 + 6A^2 - 7A - 2I = 0$

$$A^{-1} = \frac{1}{2}(-A^2 + 6A - 7I) = \frac{1}{2}\begin{bmatrix} -6 & 4 & 7 \\ 4 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$