2013 spring HW9

1.

(a)

$$u_1 = Au_0 = S\Lambda S^{-1}u_0 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1/3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 6 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 34 \\ 29 \end{bmatrix}$$

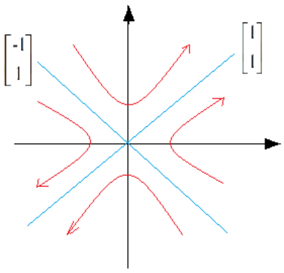
(b)

$$u_0 = Sc = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix} \Rightarrow c_1 = \frac{7}{2} \ , \ c_2 = \frac{-5}{2}$$

$$u_k = A^k u_0 = S\Lambda^k S^{-1}u_0 = S\Lambda^k c = c_1\lambda_1{}^k x_1 + c_2\lambda_2{}^k x_2 = \frac{7}{2}3^k\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{5}{2}\left(\frac{1}{3}\right)^k\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

(c) Unstable $(\because |\lambda_1| = 3 > 1)$, and origin is a saddle point.

(d)



2.

(a)

$$u_0 = Sc = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \Rightarrow c_1 = \frac{5}{2} \ , \ c_2 = \frac{-1}{2}$$

$$u(t) = c_1 e^{\lambda_1 t}x_1 + c_2 e^{\lambda_2 t}x_2 = \frac{5}{2}e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2}e^{-3t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

(b) Stable $(\because \mathrm{Re}(\lambda_1) = -1 < 0 \ , \ \mathrm{Re}(\lambda_2) = -3 < 0)$

3.

$$\det(A - \lambda_A I) = (a - \lambda_A)^2 - 1 = \lambda_A{}^2 - 2a\lambda_A + (a^2 - 1) = 0$$

$$\Rightarrow \lambda_{A1} = a + 1 \ , \ \lambda_{A2} = a - 1, \ x_{A1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, x_{A2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow u(t) = c_{A1}e^{\lambda_{A1}t}x_{A1} + c_{A2}e^{\lambda_{A2}t}x_{A2}$$

$$\because t \to \infty, u(t) \to 0 \ \therefore \lambda_{A1} = a + 1 < 0, \ \lambda_{A2} = a - 1 < 0 \Rightarrow a < -1$$

$$\det(B - \lambda_B I) = (b - \lambda_B)^2 + 1 = \lambda_B{}^2 - 2b\lambda_B + (b^2 + 1) = 0$$

$$\Rightarrow \lambda_{B1} = b - i \ , \ \lambda_{B2} = b + i, \ x_{A1} = \begin{bmatrix} 1 \\ i \end{bmatrix}, x_{A2} = \begin{bmatrix} -1 \\ i \end{bmatrix} \Rightarrow v(t) = c_{B1}e^{\lambda_{B1}t}x_{B1} + c_2 e^{\lambda_{B2}t}x_{B2}$$

$$\because t \to \infty, v(t) \to 0 \ \therefore \mathrm{Re}(\lambda_{B1}) = \mathrm{Re}(\lambda_{B2}) = b < 0 \Rightarrow b < 0$$

4.

(a)

Suppose e^{x} is the inverse of e^{At}

$e^{At}e^{x} = e^{At+x} = e^{0}I = I \Rightarrow At + x = 0 \Rightarrow x = -At$

$\therefore$ inverse of e^{At} is e^{-At}

(b)

$\because$ $Ax = \lambda x$ $\therefore e^{At}x = e^{\lambda t}x$, $e^{\lambda t} \neq 0$

5.

(a)

$$(\cos A)x = \left(I - \frac{A^2}{2!} + \frac{A^4}{4!} - \dots \right) x = x - \frac{A^2}{2!}x + \frac{A^4}{4!}x - \dots$$

$$= x - \frac{\lambda^2}{2!}x + \frac{\lambda^4}{4!}x - \dots = \left(I - \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} - \dots \right) x = (\cos\lambda)x$$

$\Rightarrow$ eigenvalue of cosA is $\cos\lambda$

(b)

$$\begin{bmatrix} \pi - \lambda_1 & \pi \\ \pi & \pi - \lambda_1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 \Rightarrow \lambda_1 = 2\pi \ , \ \begin{bmatrix} \pi - \lambda_2 & \pi \\ \pi & \pi - \lambda_2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0 \Rightarrow \lambda_2 = 0$$

$\because$ we know eigenvalue of cosA is $\cos\lambda$

$$\therefore C = \cos A = S \Lambda S^{-1} = S \begin{bmatrix} \cos\lambda_1 & 0 \\ 0 & \cos\lambda_2 \end{bmatrix} S^{-1} = SIS^{-1} = I$$

6.

$$\begin{vmatrix} 4-\lambda & -2 \\ 1 & 6-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 26 = 0 \Rightarrow \lambda = 5 \pm i \ \Rightarrow C = \begin{bmatrix} 5 & -1 \\ 1 & 5 \end{bmatrix}$$

choose $\lambda = 5 - i$, $x = \begin{bmatrix} -1-i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} i \Rightarrow S = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$

$$\begin{array}{ccc} x & \xrightarrow{A} & Ax \\ S^{-1}\downarrow & & \uparrow S \\ [x]_s & \xrightarrow{C} & C[x]_s = [Ax]_s \end{array}$$

7.

$$Ax = A(\mathrm{Re}(x) + i\,\mathrm{Im}(x)) = A\,\mathrm{Re}(x) + iA\,\mathrm{Im}(x) = \lambda x = (a - bi)(\mathrm{Re}(x) + i\,\mathrm{Im}(x))$$

$$= (a\,\mathrm{Re}(x) + b\,\mathrm{Im}(x)) + i(a\,\mathrm{Im}(x) - b\,\mathrm{Re}(x))$$

$$\mathrm{Re}\{Ax\} = (a\,\mathrm{Re}(x) + b\,\mathrm{Im}(x)) = A\,\mathrm{Re}\{x\}$$

$$\mathrm{Im}\{Ax\} = (a\,\mathrm{Im}(x) - b\,\mathrm{Re}(x)) = A\,\mathrm{Im}\{x\}$$