Linear Algebra

Problem Set 9 Solution **Spring 2015**

1.(15pts)

(a)

Let det(A- λ I) = 0 $\Rightarrow$ λ = 1, 4

$$\lambda = 1, \quad A-\lambda I = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}, \quad x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda = 4, \quad A-\lambda I = \begin{bmatrix} 0 & 3 \\ 0 & -3 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$u(t) = c_1 e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad u(0) = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \end{bmatrix}, c_1 = 2, c_2 = 3$$

$$u(t) = 2e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 3e^{4t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(b)

unstable, t $\rightarrow \infty$, u(t) $\rightarrow \infty$

(c)

By C-H, let $e^{At} = \alpha A + \beta I$

$$e^{\lambda t} = \alpha\lambda + \beta I$$

$$e^{1t} = \alpha + \beta$$

$$e^{4t} = 4\alpha + \beta$$

$$3\alpha = (4\alpha + \beta) - (\alpha + \beta) = e^{4t} - e^{1t}$$

$$e^{At} = \alpha A + \beta I = \begin{bmatrix} 4\alpha & 3\alpha \\ 0 & \alpha \end{bmatrix} + \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix} = \begin{bmatrix} 4\alpha + \beta & 3\alpha \\ 0 & \alpha + \beta \end{bmatrix} = \begin{bmatrix} e^{4t} & e^{4t} - e^{1t} \\ 0 & e^{1t} \end{bmatrix}$$

2.(15pts)

(a)

Suppose e^x is the inverse of e^{At}

$e^{At}e^x = e^{At+x} = e^0 I$, At+x = 0 $\Rightarrow$ x = -At

$\Rightarrow$ inverse of $e^{At} = e^x = e^{-At}$

$\because$ Ax = λ x $\therefore$ $e^{At}x = e^{\lambda t}x$, $e^{\lambda t} \neq 0$

(b)

By C-H, let $Q = g(A) = e^{At} = a_1 A^n + a_2 A^{n-1} + \cdots + a_n A^0$

$$\Rightarrow \quad Q^T = (e^{At})^T = a_1 (A^n)^T + a_2 (A^{n-1})^T + \cdots + a_n (A^0)^T$$

$$= a_1 (A^T)^n + a_2 (A^T)^{n-1} + \cdots + a_n (A^T)^0$$

$$= a_1 (-A)^n + a_2 (-A)^{n-1} + \cdots + a_n (-A)^0$$

$$= \mathrm{g}(-A) = e^{-At}$$

$$Q^T Q = e^{-At} e^{At} = e^{(-At+At)} = e^{0t} = I$$

3.(15pts)

(a)

$$(\cos A)x = \left(I - \frac{A^2}{2!} + \frac{A^4}{4!} + \cdots\right)x = x - \frac{A^2}{2!}x + \frac{A^4}{4!}x - \cdots = x - \frac{\lambda^2}{2!}x + \frac{\lambda^4}{4!}x - \cdots$$

$$= \left(I - \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} - \cdots\right)x = (\cos\lambda)x \quad \Rightarrow \text{eigenvalue of cosA is } \cos\lambda$$

(b)

Let det(A- λ I) = 0 $\Rightarrow$ λ = 0, 2 $x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

By C-H, let cosA = $\alpha A + \beta I$, $\cos\lambda = \alpha\lambda + \beta$

cos0 = 1 = β , cos2 = $2\alpha + \beta$, $\alpha = \dfrac{\cos 2 - 1}{2}$

$$\cos A = \alpha A + \beta I = \begin{bmatrix} \alpha & \alpha \\ \alpha & \alpha \end{bmatrix} + \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix} = \begin{bmatrix} \alpha+\beta & \alpha \\ \alpha & \alpha+\beta \end{bmatrix} = \begin{bmatrix} \frac{\cos 2 + 1}{2} & \frac{\cos 2 - 1}{2} \\ \frac{\cos 2 - 1}{2} & \frac{\cos 2 + 1}{2} \end{bmatrix}$$

4.(15pts)

(a)

Let det(A-λ I) = 0 $\Rightarrow \lambda^2 - 4\lambda + 5 = 0 \Rightarrow (\lambda - 2)^2 = 1 \Rightarrow \lambda = 2 \pm i$

$$\Rightarrow C = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

choose $\lambda = 2 - i$, $A - \lambda I = \begin{bmatrix} -1+i & -2 \\ 1 & 1+i \end{bmatrix}$, $x = \begin{bmatrix} -1-i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} i$

$$\Rightarrow S = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{array}{ccc} \text{x} & \xrightarrow{A} & \text{Ax} \\ S^{-1}\downarrow & & \uparrow S \\ [\text{x}]_s & \xrightarrow{C} & \text{C}[\text{x}]_s = [\text{Ax}]_s \end{array}$$

(b)

Let det(A-λ I) = 0 $\Rightarrow \lambda^2 - 2a\lambda + a^2 + b^2 = 0 \Rightarrow \lambda = a \pm bi$

$\lambda = a + bi$, A-λ I = $\begin{bmatrix} -bi & -b \\ b & -bi \end{bmatrix}$, $x_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

$\lambda = a - bi$, A-λ I = $\begin{bmatrix} bi & -b \\ b & bi \end{bmatrix}$, $x_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$$u(t) = c_1 e^{(a+bi)t} \begin{bmatrix} i \\ 1 \end{bmatrix} + c_2 e^{(a-bi)t} \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$\Rightarrow \text{Re}(a+bi) < 0, \quad \text{Re}(a-bi) < 0$

$\Rightarrow$ a < 0

5.(20pts)

(a)

Let det(A-λ I) = 0$\Rightarrow$ λ = 1, 2 $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow$ M=$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$A = MBM^{-1} \Rightarrow B = M^{-1}AM$

(b)

Let det(B-λ I) = 0$\Rightarrow$ λ = 0, 1 $x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow$ M'=$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

$B = M'AM'^{-1} = M^{-1}AM$

(c)

Let det(A-λ I) = 0$\Rightarrow$ λ = 2, 0 $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow$ S_A=$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

Let det(B-λ I) = 0$\Rightarrow$ λ = 2, 0 $x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow$ S_B=$\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

Let $\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, $A = S_A \Lambda S_A^{-1}$, $B = S_B \Lambda S_B^{-1}$, $S_B^{-1} B S_B = \Lambda$

$A = S_A \Lambda S_A^{-1} = S_A (S_B^{-1} B S_B) S_A^{-1} == \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{B} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \Rightarrow$ M=$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

(d)

Let det(A-λ I) = 0$\Rightarrow$ $\lambda = \lambda_1, \lambda_2$

$x_1 = \begin{bmatrix} 2 \\ \lambda_1 - 1 \end{bmatrix}, x_2 = \begin{bmatrix} \lambda_2 - 4 \\ 3 \end{bmatrix} \Rightarrow$ S_A=$\begin{bmatrix} 2 & \lambda_2 - 4 \\ \lambda_1 - 1 & 3 \end{bmatrix}$

Let det(B-λ I) = 0 $\Rightarrow x_1 = \begin{bmatrix} \lambda_1 - 1 \\ 2 \end{bmatrix}, x_2 = \begin{bmatrix} 3 \\ \lambda_2 - 4 \end{bmatrix} \Rightarrow$ S_B=$\begin{bmatrix} \lambda_1 - 1 & 3 \\ 2 & \lambda_2 - 4 \end{bmatrix}$

Let $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ $A = S_A \Lambda S_A^{-1}$, $B = S_B \Lambda S_B^{-1}$

$A = S_A \Lambda S_A^{-1} = S_A (S_B^{-1} B S_B) S_A^{-1} == \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{B} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \Rightarrow$ M=$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

6.(20pts)

(a)

False

Let A=$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, A^{-1}=$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, A is similar to A^{-1}, $A \neq I$

(b)

True

Let eigenvalue of A is λ, then eigenvalue of A+I = λ+1

A and A+I has different eigenvalue $\Rightarrow$ They are not similar

(c)

True

$AB = B^{-1}B(AB) = B^{-1}(BA)B$

(d)

True

$A = MBM^{-1} \Rightarrow A^2 = (MBM^{-1})(MBM^{-1}) = MB^2M^{-1}$

(e)

True

Let A = $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, B = $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $A^2 = B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

A^2 is similar to B^2, but A is not similar to B(different eigenvalue)