Linear Algebra

Problem Set 9 Solution **Spring 2016**

1. (15pts)

(a)

$$\frac{d(v+w)}{dt} = \frac{dv}{dt} + \frac{dw}{dt} = w - v + v - w = 0$$

$$v + w = c_1 = 50$$

(b)

$$\frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} \frac{d\boldsymbol{v}}{dt} \\ \frac{d\boldsymbol{w}}{dt} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v \\ w \end{bmatrix} = A\boldsymbol{u}$$

$$\lambda = 0, -2$$

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(c)

$$\boldsymbol{u} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$c_1 = 25, c_2 = 15$$

$$\boldsymbol{u}(1) = 25\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 15e^{-2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

2.(10pts)

$$Q = e^{At} = I + At + \frac{(At)^2}{2!} + \cdots$$

$$Q^T = I + A^T t + \frac{(A^T t)^2}{2!} + \cdots = I + (-At) + \frac{(-At)^2}{2!} + \cdots = e^{-At}$$

$$Q^T Q = e^{-At}e^{At} = I$$

3.(10pts)

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots = I + A\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) = I + A(e^t - 1)$$

$$e^{Bt} = \begin{bmatrix} e^t & 0 \\ e^t - 1 & 1 \end{bmatrix}$$

4.(30pts)

(a)False

eigenvalue of A is λ, eigenvalue of $A+I$ is $\lambda+1$

(b)True

A and B have same eigenvalues, so $\det(A) = \det(B) \neq 0$

(c)True

$A = SBS^{-1}$

$A^2 = (SBS^{-1})(SBS^{-1}) = SB^2S^{-1}$

(d)False

$A^2x = \lambda^2 x, B^2 y = \lambda^2 y$

When eigenvalue of A is λ, and B is $-\lambda$, they are not similar.

(e)True

$AB = AB(AA^{-1}) = A(BA)A^{-1}$

(f)False

$\det\left(\begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}\right) = 2 \neq \det\left(\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\right) = 1$

(g)True

$A = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}, B = \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$

$A = PBP^{-1}, P = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}$

(h)True

$A = \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}, B = \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}$

$A = PBP^{-1}, P = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$

(f)True

$A = \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}, B = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$

$A = PBP^{-1}, P = \begin{bmatrix}2 & 2\\0 & 1\end{bmatrix}$

(j)False

$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$PBP^{-1} = I \neq A$

5.(15pts)

(a)

$Ax = \lambda x, A^2x = \lambda^2 x, \cdots$

$$(cosA)x = Ix - \frac{A^2x}{2!} + \frac{A^4x}{4!} \cdots = Ix - \frac{\lambda^2 x}{2!} + \frac{\lambda^4 x}{4!} \cdots = (cos\lambda)x$$

(b)

$$\begin{bmatrix} \pi - \lambda_1 & \pi \\ \pi & \pi - \lambda_1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0, \lambda_1 = 2\pi$$

$$\begin{bmatrix} \pi - \lambda_2 & \pi \\ \pi & \pi - \lambda_2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0, \lambda_2 = 0$$

$$C = cosA = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} cos2\pi & 0 \\ 0 & cos0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = I$$

6.(10pts)

$\lambda_1 = 2 + i, \lambda_2 = 2 - i$

$x_1 = \begin{bmatrix} 2 \\ -1 - i \end{bmatrix}, x_2 = \begin{bmatrix} 2 \\ -1 + i \end{bmatrix}$ or $x_1 = \begin{bmatrix} 1 + i \\ -1 \end{bmatrix}, x_2 = \begin{bmatrix} 1 - i \\ -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}^{-1}$$

7.(10pts)

$\lambda_A = a \pm 1, \lambda_B = b \pm i$

t $\rightarrow \infty$, Re(λ_A) and Re(λ_B) $< 0 \rightarrow a < -1, b < 0$